Published by Institute of Physics Publishing for SISSA
Received: *June 27, 2005*
Accepted: *August 6, 2005*
Published: *August 31, 2005*

Generalized spacetimes defined by cubic forms and the minimal unitary realizations of their quasiconformal groups

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Abstract: We study the symmetries of generalized spacetimes and corresponding phase spaces defined by Jordan algebras of degree three. The generic Jordan family of formally real Jordan algebras of degree three describe extensions of the minkowskian spacetimes by an extra "dilatonic" coordinate, whose rotation, Lorentz and conformal groups are $\mathrm{SO}(d-1)$, $\mathrm{SO}(d-1,1)\times\mathrm{SO}(1,1)$ and $\mathrm{SO}(d,2)\times\mathrm{SO}(2,1)$, respectively. The generalized spacetimes described by simple Jordan algebras of degree three correspond to extensions of minkowskian spacetimes in the critical dimensions ($d = 3,4,6,10$) by a dilatonic and extra $(2,4,8,16)$ commuting spinorial coordinates, respectively. Their rotation, Lorentz and conformal groups are those that occur in the first three rows of the Magic Square. The Freudenthal triple systems defined over these Jordan algebras describe conformally covariant phase spaces. Following hep-th/0008063, we give a unified geometric realization of the quasiconformal groups that act on their conformal phase spaces extended by an extra "cocycle" coordinate. For the generic Jordan family the quasiconformal groups are $\mathrm{SO}(d+2,4)$, whose minimal unitary realizations are given. The minimal unitary representations of the quasiconformal groups $\mathrm{F}_{4(4)}$, $\mathrm{E}_{6(2)}$, $\mathrm{E}_{7(-5)}$ and $\mathrm{E}_{8(-24)}$ of the simple Jordan family were given in our earlier work [10].

Keywords: Space-Time Symmetries, M-Theory, Supersymmetry and Duality, Supergravity Models.

Contents

1. Introduction

One can define generalized space-times coordinatized by Jordan algebras J, in such a way that their rotation, Lorentz, and conformal groups can be identified with the automorphism, reduced structure, and the linear fractional groups of of the corresponding Jordan algebras J, respectively [1 − 4]. The main requirements for formulating relativistic quantum field theories over four dimensional Minkowski spacetime extend naturally to the generalized space times defined by formally real Jordan algebras. For example, the well-known connection between the positive energy unitary representations of the four dimensional conformal group $SU(2,2)$ and the covariant fields transforming in finite dimensional representations of the Lorentz group $SL(2,\mathbb{C})$ [5, 6] extends to all generalized space-times defined by formally real Jordan algebras [7].

Except for G_2, F_4 and E_8, certain noncompact real forms of all simple groups arise as conformal groups of formally real Jordan algebras. The corresponding real forms are precisely the ones that admit positive energy unitary representations. These conformal groups act geometrically on the corresponding Jordan algebras via fractional linear transformations [8]. In particular, the exceptional group E_7 acts as the conformal group of the exceptional Jordan algebra [1]. For the real exceptional Jordan algebra $J_3^{\mathbb{O}}$ over the division algebra of real octonions the conformal group is $E_{7(-25)}$, which admits positive energy unitary representations.

Motivated by possible applications to the U-duality groups of maximal supergravity and M-theory, the first geometric realization of $E_{8(8)}$ was given in [9]. This geometric realization of $E_{8(8)}$ on a 57 dimensional space was called quasiconformal since it leaves invariant a suitably defined light cone with respect to a quartic norm. This construction of $E_{8(8)}$ together with the corresponding construction of $E_{8(-24)}$ [10] contain all previous geometric realizations of the symmetries of generalized space-times based on exceptional Lie groups, and at the same time goes beyond the framework of Jordan algebras.

The algebraic structures related to the novel quasiconformal realizations are not Jordan algebras, but rather Freudenthal triple systems (FTS) [14, 15]. The 57 dimensional space on which $E_{8(8)}$ or $E_{8(-24)}$ is realized is the direct sum of a 56 dimensional space of the underlying FTS and a singlet defined by the symplectic invariant over that FTS. The nonlinear realization of the Lie algebra of $E_{8(8)}$ given in [9] was written in terms of the triple product of the FTS and hence extends to all simple Lie algebras since they all can be constructed over FTS's. This follows from the fact that simple Freudenthal triple systems are in one-to-one correspondence with simple Lie algebras with a five graded decomposition

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} . \tag{1.1}$$

such that grade ± 2 spaces are one dimensional [16]. These include all simple Lie algebras, except for $\mathfrak{sl}(2)$ for which it degenerates to a 3-graded structure.

The formally real Jordan algebras of degree three and their associated geometries arise naturally within the framework of $N = 2$ Maxwell-Einstein supergravity theories (MESGT) in five dimensions [17]. The geometries and symmetries of the corresponding four dimensional MESGT's are deeply related to the FTS's defined over these Jordan algebras.[1] The formally real Jordan algebras of degree three and the corresponding FTS's were also shown to be related to relativistic point particle and classical bosonic string actions by Sierra [18].

In this paper we will study the quasiconformal groups associated with FTS's defined over formally real Jordan algebras[2] a la Freudenthal. Such FTS's were classified by Ferrar [19] who showed that FTS's defined over Jordan algebras require that the Jordan

[1]For a review and the references on the subject see [27].

[2]Formally real or euclidean Jordan algebras J are such that $a^2 + b^2 = 0$ implies that both $a = 0$ and $b = 0$ for all $a, b \in J$.

algebras be of degree three. The generic formally real Jordan algebras of degree three define spacetimes which can be interpreted as extensions of minkowskian spacetimes by an extra dilatonic coordinate. Their Lorentz groups are of the form

$$\mathrm{SO}(d-1,1)\times\mathrm{SO}(1,1) \tag{1.2}$$

and their conformal groups are

$$\mathrm{SO}(d,2)\times\mathrm{SO}(2,1)\,. \tag{1.3}$$

The spacetimes defined by four simple formally real Jordan algebras of degree three describe extensions of the *critical* minkowskian spacetimes in $d=3,4,6,10$ dimensions by a dilatonic *and* $2,4,8,16$ spinorial commuting coordinates, respectively. Rotation groups of these space-times are

$$\mathrm{SO}\,(3)\,,\quad \mathrm{SU}\,(3)\,,\quad \mathrm{USp}\,(6)\,,\quad \mathrm{F}_4\,. \tag{1.4}$$

Their Lorentz groups are

$$\mathrm{SL}\,(3,\mathbb{R})\,,\quad \mathrm{SL}\,(3,\mathbb{C})\,,\quad \mathrm{SU}^*\,(6)\,,\quad \mathrm{E}_{6(-26)} \tag{1.5}$$

and their conformal groups are

$$\mathrm{Sp}\,(6,\mathbb{R})\,,\quad \mathrm{SU}\,(3,3)\,,\quad \mathrm{SO}^*\,(12)\,,\quad \mathrm{E}_{7(-25)} \tag{1.6}$$

respectively. Hence their rotation, Lorentz and conformal groups coincide with the first three rows of the Magic Square [21].

The quasiconformal groups associated with the spacetimes defined by the generic Jordan algebras of degree three are

$$\mathrm{SO}\,(d+2,4) \tag{1.7}$$

while the quasiconformal groups associated with the spacetimes defined by simple Jordan algebras of degree are

$$\mathrm{F}_{4(4)}\,,\quad \mathrm{E}_{6(2)}\,,\quad \mathrm{E}_{7(-5)}\,,\quad \mathrm{E}_{8(-24)}\,. \tag{1.8}$$

One of the remarkable features of the quasiconformal realizations of noncompact groups is that their quantization leads naturally to the minimal unitary representations of the respective noncompact groups [28, 10, 27]. The concept of a minimal unitary representation of a non-compact group G was first introduced by A. Joseph [29]. Over the last two decades there has been a great deal of work done by the mathematicians on the minimal unitary representations of noncompact groups. For references on earlier work on the subject we refer the reader to the review lectures of Jian-Shu Li [11]. More recently, minimal unitary representations were studied by Kazhdan, Pioline, and Waldron [12], by Günaydin, Koepsell and Nicolai [28] and by Günaydin and Pavlyk [10]. The construction of KPW extends to all simply laced groups and was motivated by the idea that the theta series of $\mathrm{E}_{8(8)}$ and its subgroups may describe the quantum supermembrane in various dimensions [13]. One of the main motivations of the work of GKN as well as of GP was the idea that

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the spectra of M-theory in various dimensions must fall into unitary representations of its U-duality groups in the respective dimensions. Furthermore the U-duality groups in 3 and 4 dimensions act as spectrum generating symmetry groups in the charge space of BPS black hole solutions in 4 and 5 dimensions, respectively [9, 27]. Realization of the minimal unitary representation of $\mathrm{E}_{8(8)}$ and its subalgebras given in [28] is based on their geometric realizations as quasi-conformal groups. In our earlier work [10] we constructed the minimal unitary representation of the other noncompact real form of E_8, namely $E_{8(-24)}$, and those of its subgroups, that arise as U-duality groups of MESGT's defined by simple Jordan algebras of degree three. These minimal unitary representations correspond to quantizations of their geometric realizations as quasiconformal groups as well. In this paper we will study the quasiconformal groups associated with all euclidean Jordan algebras of degree three mainly from a space-time point of view. We will also give the minimal unitary realizations of the quasiconformal groups $\mathrm{SO}(d+2,4)$ of spacetimes defined by the generic Jordan family.

The plan of the paper is as follows. We review U-duality groups arising in $N=2$ Maxwell-Einstein supergravity theories (MESGT) defined by cubic forms in dimensions 5,4 and 3 in section 2. In section 3 we review generalized space-times defined by Jordan algebras as well as their symmetry groups. In section 4 we study the spacetimes defined by Jordan algebras of degree 3 as extensions of minkowskian space-times with dilatonic and spinorial coordinates and their symmetries. We the review the realization of the quasiconformal groups via the Freudenthal triple product given in [9]. In section 5 we present the explicit geometric realizations of quasi-conformal groups $\mathrm{SO}\,(d+2,4)$ of the generic Jordan family of spacetimes. In section 6 we give an explicit geometric realization of $\mathrm{E}_{8(-24)}$ as a quasiconformal group as the aforementioned extension of $\mathrm{SO}\,(12,4)$. By consistent truncation of $E_{8(-24)}$ we obtain the geometric quasiconformal realizations of $E_{7(-5)}, E_{6(2)}$ and $F_{4(4)}$, which we give explicitly as well. In the final section we give the minimal unitary realizations of the generic family of quasiconformal groups $\mathrm{SO}\,(d+2,4)$. Together with our earlier results given in [10] this completes the construction of the minimal unitary representations of quasiconformal groups defined over euclidean Jordan algebras of degree three by quantizations of their geometric realizations.

2. U-duality groups of $N=2$ Maxwell-Einstein supergravity theories in $d=5,4,3$ dimensions defined by cubic forms

2.1 Five dimensional $N=2$ MESGT's

In this section we will review the geometry and the symmetry groups of $N=2$ MESGT's in five dimensions and the corresponding dimensionally reduced theories in $d=4$ and $d=3$ dimensions. The MESGT's describe the coupling of an arbitrary number n of (abelian) vector fields to $N=2$ supergravity and five dimensional MESGT's were constructed in [17]. The bosonic part of the lagrangian can be written as [17]

$$e^{-1}\mathcal{L}_{\text{bosonic}} = -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F^{I}_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}g_{xy}(\partial_\mu\varphi^x)(\partial^\mu\varphi^y) + \frac{e^{-1}}{6\sqrt{6}}C_{IJK}\varepsilon^{\mu\nu\rho\sigma\lambda}F^{I}_{\mu\nu}F^{J}_{\rho\sigma}A^{K}_{\lambda}\,, \tag{2.1}$$

where e and R denote the fünfbein determinant and the scalar curvature in $d=5$, respectively. $F^I_{\mu\nu}$ are the field strengths of the abelian vector fields A^I_μ, $(I=0,1,2\ldots,n)$ with A^0_μ denoting the "bare" graviphoton. The metric, g_{xy}, of the scalar manifold $\mathcal{M}$ and the "metric" $\overset{\circ}{a}_{IJ}$ of the kinetic energy term of the vector fields both depend on the scalar fields φ^x ($x,y,\ldots=1,2,\ldots,n$). The invariance under abelian gauge transformations of the vector fields requires the completely symmetric tensor C_{IJK} to be constant. Remarkably, one finds that the entire $N=2$, $d=5$ MESGT is uniquely determined by the constant tensor C_{IJK} [17]. In particular, the metrics of the kinetic energy terms of the vector and scalar fields are determined by C_{IJK}. More specifically, consider the cubic polynomial, $\mathcal{V}(h)$, in $(n+1)$ real variables h^I $(I=0,1,\ldots,n)$ defined by the C_{IJK}

$$\mathcal{V}(h) := C_{IJK}\, h^I h^J h^K\,. \tag{2.2}$$

Using this polynomial as a real "Kähler potential" for a metric, a_{IJ}, in an $n{+}1$ dimensional ambient space with the coordinates h^I:

$$a_{IJ}(h) := -\frac{1}{3}\frac{\partial}{\partial h^I}\frac{\partial}{\partial h^J}\ln \mathcal{V}(h)\,. \tag{2.3}$$

one finds that the n-dimensional target space, $\mathcal{M}$, of the scalar fields φ^x can be identified with the hypersurface [17]

$$\mathcal{V}(h) = C_{IJK}h^I h^J h^K = 1 \tag{2.4}$$

in this space. The metric g_{xy} of the scalar manifold is simply the pull-back of (2.3) to $\mathcal{M}$

$$g_{xy} = h^I_x h^J_y \overset{\circ}{a}_{IJ} \tag{2.5}$$

where

$$h^I_x = -\sqrt{\frac{3}{2}}\frac{\partial}{\partial \phi^x} h^I \tag{2.6}$$

and one finds that the Riemann curvature of the scalar manifold has the simple form

$$K_{xyzu} = \frac{4}{3}\left(g_{x[u}g_{z]y} + T_{x[u}{}^{w}T_{z]yw}\right) \tag{2.7}$$

where T_{xyz} is the symmetric tensor

$$T_{xyz} = h^I_x h^J_y h^K_z C_{IJK}\,. \tag{2.8}$$

The "metric" $\overset{\circ}{a}_{IJ}(\varphi)$ of the kinetic energy term of the vector fields appearing in (2.1) is given by the componentwise restriction of a_{IJ} to $\mathcal{M}$:

$$\overset{\circ}{a}_{IJ}(\varphi) = a_{IJ}|_{\mathcal{V}=1}\,. \tag{2.9}$$

We should stress that the indices $I,J,K,\ldots$ are lowered and raised by the metric $\overset{\circ}{a}_{IJ}(\varphi)$ and its inverse. The physical requirement of positivity of kinetic energy requires that g_{xy} and $\overset{\circ}{a}_{IJ}$ be positive definite metrics. This requirement induces constraints on the possible

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C_{IJK}, and in [17] it was shown that any C_{IJK} that satisfy these constraints can be brought to the following form

$$C_{000} = 1\,, \quad C_{0ij} = -\frac{1}{2}\delta_{ij}\,, \quad C_{00i} = 0\,, \tag{2.10}$$

with the remaining coefficients C_{ijk} $(i, j, k = 1, 2, \ldots, n)$ being completely arbitrary. This basis is referred to as the canonical basis for C_{IJK}.

Denoting the symmetry group of the tensor C_{IJK} as G one finds that the full symmetry group of $N = 2$ MESGT in $d = 5$ is of the form $G \times \mathrm{SU}(2)_R$ where $\mathrm{SU}(2)_R$ denotes the local R-symmetry group of the $N = 2$ supersymmetry algebra.

2.2 Symmetric target spaces and Jordan algebras

From the form of the Riemann curvature tensor K_{xyzu} it is clear that the covariant constancy of T_{xyz} implies the covariant constancy of K_{xyzu}:

$$T_{xyz;w} = 0 \implies K_{xyzu;w} = 0\,.$$

Therefore the scalar manifolds $\mathcal{M}_5$ with covariantly constant T tensor are locally symmetric spaces.

If $\mathcal{M}_5$ is a homogeneous space the covariant constancy of T_{xyz} was shown to be equivalent to the following identity [17]:

$$C^{IJK} C_{J(MN} C_{PQ)K} = \delta^I{}_{(M} C_{NPQ)} \tag{2.11}$$

where the indices are raised by $\overset{\circ}{a}{}^{IJ}$.[3]

Remarkably the cubic forms defined by C_{IJK} of the $N = 2$ MESGT's with $n \geq 2$ with a symmetric target space $\mathcal{M}_5$ and a covariantly constant T tensor are in one-to-one correspondence with the norm forms of euclidean (formally real) Jordan algebras of degree 3.

The precise connection between Jordan algebras of degree 3 and the geometries of MESGT's with symmetric target spaces in $d = 5$ was established [17] through a novel formulation of the corresponding Jordan algebras. This formulation is due to McCrimmon [20], who generalized and unified previous constructions by Freudenthal, Springer and Tits [21], which we outline here.

Let V be a vector space over the field of reals $\mathbb{R}$, and let $\mathcal{V}\colon V \times V \times V \to \mathbb{R}$ be a cubic norm on V. Furthermore, assume that there exists a quadratic map $\sharp : x \to x^\sharp$ of V into itself and a "base point" $c \in V$ such that

$$\mathcal{V}\,(c) = 1 \qquad \text{and} \qquad c^\sharp = c \tag{i), (ii}$$

$$T\left(x^\sharp, y\right) = y^I \partial_I \mathcal{V}\big|_x \tag{iii}$$

[3]For proof of this equivalence an expression for constants C_{IJK} in terms of scalar field dependent quantities was used

$$C_{IJK} = \frac{5}{2} h_I h_K h_K - \frac{3}{2} \overset{\circ}{a}_{(IJ} h_{K)} + T_{xyz} h_I^x h_J^y h_K^z$$

as well as algebraic constraints $h_I h^I = 1$ and $h_x^I h_I = 0$ that follows from susy.

$$c \times y = T(y,c)\, c - y \tag{iv}$$

$$\left(x^\sharp\right)^\sharp = \mathcal{V}(x)\, x \,. \tag{v}$$

The last equation is referred to as the adjoint identity. The map $T : V \times V \rightarrow \mathbb{R}$ is defined as

$$T(x,y) = -\, x^I y^J \partial_I \partial_J \ln \mathcal{V}\big|_c \tag{2.12a}$$

and the Freudenthal product $\times$ of two elements x and y is defined as

$$x \times y = (x+y)^\sharp - x^\sharp - y^\sharp \,. \tag{2.12b}$$

McCrimmon showed that a vector space with the above properties defines a unital Jordan algebra with Jordan product $\circ$ given by

$$x \circ y = \frac{1}{2}\Big(T(c,x)\, y + T(c,y)\, x - T(c, x \times y)\, c + x \times y \Big) \tag{2.13}$$

and a quadratic operator U_x given by

$$U_x y = T(x,y)\, x - x^\sharp \times y \,. \tag{2.14}$$

In [17] it was shown that the properties (i) and (iv) are satisfied by the cubic norm form defined by the tensor C_{IJK} of $N = 2$ MESGT's in $d = 5$. The condition of adjoint identity is equivalent to the requirement that the scalar manifold be symmetric space with a covariantly constant T-tensor [17]. The corresponding symmetric spaces are of the form

$$\mathcal{M} = \frac{\mathrm{Str}_0(J)}{\mathrm{Aut}(J)} \tag{2.15}$$

where $\mathrm{Str}_0(J)$ and $\mathrm{Aut}(J)$ are the reduced structure group and automorphism group of the Jordan algebra J respectively.

From the foregoing we see that the classification of locally symmetric spaces $\mathcal{M}$ for which the tensor T_{xyz} is covariantly constant reduces to the classification of Jordan algebras with cubic norm forms. Following Schafers [23] the possibilities were listed in [17]:

1. $J = \mathbb{R}$, $\mathcal{V}(x) = x^3$. The base point may be chosen as $c = 1$. This case supplies $n = 0$, i.e. pure $d = 5$ supergravity.

2. $J = \mathbb{R} \oplus \Gamma$, where Γ is a simple algebra with identity $\mathbf{e}_2$ and quadratic norm $Q(\mathbf{x})$, for $\mathbf{x} \in \Gamma$, such that $Q(\mathbf{e}_2) = 1$. The norm is given as $\mathcal{V}(x) = aQ(\mathbf{x})$, with $x = (a, \mathbf{x})$. The base point may be chosen as $c = (1, \mathbf{e}_2)$. This includes two special cases

   (a) $\Gamma = \mathbb{R}$ and $Q = b^2$, with $\mathcal{V} = ab^2$. This is applicable to $n = 1$.

   (b) $\Gamma = \mathbb{R} \oplus \mathbb{R}$ and $Q = bc$, and $\mathcal{V} = abc$ and is applicable to $n = 2$.

   Notice that for these special cases the norm is completely factorized, so that the space $\mathcal{C}$ and therefore $\mathcal{M}$, is flat. For $n > 2$, $\mathcal{V}$ is still factorized into a linear and quadratic parts, so that $\mathcal{M}$ is still reducible. The positive definiteness of the metric a_{IJ} of $\mathcal{C}$, which is required on the physical grounds, requires that Q have Minkowski signature $(+,-,-,\dots,-)$. The point $\mathbf{e}_2$ can be chosen as $(1,0,\dots,0)$. It is then obvious that the invariance group of the norm is

$$\mathrm{Str}_0\,(J) = \mathrm{SO}\,(n-1,1) \times \mathrm{SO}\,(1,1) \tag{2.16}$$

where the SO $(1,1)$ factor arises from the invariance of $\mathcal{V}$ under the dilatation $(a,\mathbf{x}) \to \left(e^{-2\lambda}a, e^{\lambda}\mathbf{x}\right)$ for $\lambda \in \mathbb{R}$, and that SO $(n-1)$ is Aut (J). Hence

$$\mathcal{M} = \frac{\mathrm{SO}\,(n-1,1)}{\mathrm{SO}\,(n-1)} \times \mathrm{SO}\,(1,1)\,. \tag{2.17}$$

3. Simple euclidean Jordan algebras $J = J_3^{\mathbb{A}}$ generated by 3×3 hermitean matrices over the four division algebras $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. In these four cases an element $x \in J$ can be written as

$$x = \begin{pmatrix} \alpha_1 & a_3 & a_2^* \\ a_3^* & \alpha_2 & a_1 \\ a_2 & a_1^* & \alpha_3 \end{pmatrix} \tag{2.18}$$

where $\alpha_k \in \mathbb{R}$ and $a_k \in \mathbb{A}$ with $*$ indicating the conjugation in the underlying dvision algebra. The cubic norm $\mathcal{V}$, following Freudenthal [21], is given by

$$\mathcal{V}\,(x) = \alpha_1\alpha_2\alpha_3 - \alpha_1 \left|a_1\right|^2 - \alpha_2 \left|a_2\right|^2 - \alpha_3 \left|a_3\right|^2 + a_1a_2a_3 + \left(a_1a_2a_3\right)^*\,. \tag{2.19}$$

For $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$ it coincides with the usual definition of determinant $Det(x)$. The corresponding spaces $\mathcal{M}$ are irreducible of dimension $3\,(1+\dim\mathbb{A})-1$, which we list below:

$$\begin{aligned} \mathcal{M}(J_3^{\mathbb{R}}) &= \frac{\mathrm{SL}\,(3,\mathbb{R})}{\mathrm{SO}\,(3)} \qquad & \mathcal{M}(J_3^{\mathbb{H}}) &= \frac{\mathrm{SU}^*\,(6)}{\mathrm{USp}\,(6)} \\ \mathcal{M}(J_3^{\mathbb{C}}) &= \frac{\mathrm{SL}\,(3,\mathbb{C})}{\mathrm{SU}\,(3)} \qquad & \mathcal{M}(J_3^{\mathbb{O}}) &= \frac{\mathrm{E}_{6(-26)}}{\mathrm{F}_4}\,. \end{aligned} \tag{2.20}$$

The magical supergravity theories described by simple Jordan algebras $J_A^{\mathbb{A}}$ (A = R, C, H or O) can be truncated to theories belonging to the generic families. This is achieved by restricting the elements of $J_A^{\mathbb{A}}$

$$\begin{pmatrix} \alpha_1 & a_3 & \overline{a}_2 \\ \overline{a}_3 & \alpha_2 & a_1 \\ a_2 & \overline{a}_1 & \alpha_3 \end{pmatrix} \tag{2.21}$$

to lie in their subalgebra $J = \mathbb{R} \oplus J_2^{\mathbb{A}}$ be setting $a_1 = a_2 = 0$. Their symmetry groups are as follows:

$$\begin{aligned} J = \mathbb{R} \oplus J_2^{\mathbb{R}} &: \ \mathrm{SO}(1,1) \times \mathrm{SO}\,(2,1) \subset \mathrm{SL}\,(3,\mathbb{R}) \\ J = \mathbb{R} \oplus J_2^{\mathbb{C}} &: \ \mathrm{SO}(1,1) \times \mathrm{SO}\,(3,1) \subset \mathrm{SL}\,(3,\mathbb{C}) \\ J = \mathbb{R} \oplus J_2^{\mathbb{H}} &: \ \mathrm{SO}(1,1) \times \mathrm{SO}\,(5,1) \subset \mathrm{SU}^*\,(6) \\ J = \mathbb{R} \oplus J_2^{\mathbb{O}} &: \ \mathrm{SO}(1,1) \times \mathrm{SO}\,(9,1) \subset \mathrm{E}_{6(-26)}\,. \end{aligned} \tag{2.22}$$

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2.3 Geometries of the four dimensional MESGTs defined by Jordan algebras of degree 3

Under dimensional reduction to the $4D$ the kinetic energy of the scalar fields of the $5D$ $N = 2$ MESGTs can be written as [17]

$$e^{-1}\mathcal{L}_{\text{scalars}} = -g_{IJ}\partial_\mu Z^I \partial^\mu \overline{Z}^J \tag{2.23}$$

where

$$g_{IJ} = \hat{a}_{IJ}\left(Z - \overline{Z}\right) = -\frac{1}{2}\frac{\partial}{\partial Z^I}\frac{\partial}{\partial \overline{Z}^J}\ln \mathcal{V}\left(Z - \overline{Z}\right) \tag{2.24}$$

and Z^I are complex scalar fields

$$Z^I = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}A^I + i\hat{h}^I\right) \tag{2.25}$$

where the real parts A^I are scalars coming from the vectors in 5 dimensions and $\hat{h}^I$ are

$$\hat{h}^I = e^\sigma h^I\left(\phi^x\right) \tag{2.26}$$

where σ is the scalar coming from the graviton in the five dimensions. Since $\mathcal{V}\left(\hat{h}\right) = e^{3\sigma} > 0$ the scalar manifold in $4D$ theories corresponds to the "upper half-plane" with respect to the cubic norm. For euclidean Jordan algebras of degree three these are the Koecher upper half-spaces [8] of the corresponding Jordan algebras

$$\mathcal{M}_4 = \mathcal{D}\left(J\right) = J + i\mathcal{C}\left(J\right) \tag{2.27}$$

where $\mathcal{C}(J)$ denotes elements of the Jordan algebra with positive cubic norm. The Koecher half-spaces are bi-holomorphically equivalent to bounded symmetric domains whose Bergmann kernel is simply $\mathcal{V}(Z - \overline{Z})$. As was first shown in [26] the scalar manifold of the $4D$ MESGTs must be special Kähler. For the theories coming from $5D$ the Kähler potential reads

$$F\left(Z, \overline{Z}\right) = -\frac{1}{2}\ln \mathcal{V}\left(Z - \overline{Z}\right) \tag{2.28}$$

and are called very special Kähler geometries.

The bounded symmetric domains associated with the upper half-spaces of Jordan algebras are isomorphic to certain hermitian symmetric spaces. For the euclidean Jordan algebras of degree 3 these spaces are as follows:

$$\begin{aligned}
\mathcal{M}_4\left(J = \mathbb{R} + \Gamma\left(Q\right)\right) &= \frac{\mathrm{SO}\left(2,1\right) \times \mathrm{SO}\left(n,2\right)}{\mathrm{SO}(2) \times \mathrm{SO}\left(n\right) \times \mathrm{SO}(2)} \\
\mathcal{M}_4\left(J_3^{\mathbb{R}}\right) &= \frac{\mathrm{Sp}\left(6,\mathbb{R}\right)}{\mathrm{U}(3)} \\
\mathcal{M}_4\left(J_3^{\mathbb{C}}\right) &= \frac{\mathrm{SU}\left(3,3\right)}{\mathrm{S}\left(\mathrm{U}(3) \times \mathrm{U}(3)\right)} \\
\mathcal{M}_4\left(J_3^{\mathbb{H}}\right) &= \frac{\mathrm{SO}^*\left(12\right)}{\mathrm{U}(6)}
\end{aligned}$$

$$\mathcal{M}_4\left(J_3^{\mathbb{O}}\right) = \frac{\mathrm{E}_{7(-25)}}{\mathrm{E}_6 \times \mathrm{U}(1)}\,. \tag{2.29}$$

These symmetric spaces are simply the quotients of the conformal groups of the corresponding Jordan algebras by their maximal compact subgroups:

$$\mathcal{M}_4 = \frac{\mathrm{Conf.}\,(J)}{K\,(J)}$$

The correspondence between the vector fields and the elements of the underlying Jordan algebras in five dimensions gets extended to a correspondence between the vector field strengths $F^A_{\mu\nu}$ plus their magnetic duals $G^A_{\mu\nu}$ with the elements of the Freudenthal triple system defined by the Jordan algebra of degree three

$$F^A_{\mu\nu} \oplus G^A_{\mu\nu} \Leftrightarrow \mathcal{FTS}(J)\,. \tag{2.30}$$

The automorphism group of this FTS is isomorphic to the four dimensional U-duality group and it acts as the spectrum generating conformal group on the charge space of the BPS black hole solutions of five dimensional MESGT's [9, 27].

2.4 Geometries of the three dimensional MESGTs defined by Jordan algebras of degree 3

Upon further dimensional reduction to 3 space-time dimensions, the MESGTs defined by euclidean Jordan algebras of degree three have target spaces that are quaternionic symmetric spaces. The corresponding symmetric spaces are:

$$\begin{gathered}
\mathcal{M}_3\left(J = \mathbb{R} + \Gamma\,(Q)\right) = \frac{\mathrm{SO}(n+2,4)}{\mathrm{SO}(n+2)\times\mathrm{SO}(4)} \\
\mathcal{M}_3\left(J_3^{\mathbb{R}}\right) = \frac{\mathrm{F}_{4(4)}}{\mathrm{USp}(6)\times\mathrm{SU}(2)} \qquad \mathcal{M}_3\left(J_3^{\mathbb{H}}\right) = \frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12)\times SU(2)} \\
\mathcal{M}_3\left(J_3^{\mathbb{C}}\right) = \frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)} \qquad \mathcal{M}_3\left(J_3^{\mathbb{O}}\right) = \frac{\mathrm{E}_{8(-24)}}{\mathrm{E}_7\times\mathrm{SU}(2)}
\end{gathered} \tag{2.31}$$

The pure $5d$, $N=2$ supergravity under dimensional reduction to three dimensions leads to the target space

$$\frac{\mathrm{G}_{2(2)}}{\mathrm{SU}(2)\times\mathrm{SU}(2)} \tag{2.32}$$

which can be embedded in the coset space

$$\frac{\mathrm{SO}(3,4)}{\mathrm{SO}(3)\times\mathrm{SO}(4)}\,. \tag{2.33}$$

We should note that the above target spaces are obtained after dualizing all the bosonic propagating fields to scalar fields which is special to three dimensions. The Lie algebras of the three dimensional U-duality groups have a 5-graded decomposition with respect to the four dimensional U-duality groups. They are isomorphic to the quasiconformal groups constructed over the corresponding FTS's, which act as spectrum generating symmetry group on the charge-entropy space of BPS black hole solutions in four dimensional MESGT's [9, 27].

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3. Generalized space-times defined by Jordan algebras and their symmetry groups

3.1 Generalized rotation, Lorentz and conformal groups

In the previous sections we reviewed how Jordan algebras arise in a fundamental way within the framework of supergravity theories. In this section we will review work on how Jordan algebras can be used to generalize four dimensional Minkowski spacetime and its symmetry groups in a natural way. The first proposal to use Jordan algebras to define generalized spacetimes was made in the early days of spacetime supersymmetry in attempts to find the super analogs of the exceptional Lie algebras [1]. One of the main anchors of this proposal was the twistor formalism, which in four-dimensional space-time ($d = 4$) leads naturally to the representation of four vectors in terms of 2×2 hermitean matrices over the field of complex numbers $\mathbb{C}$. In particular, a coordinate four-vector x_μ can be represented as:

$$x = x_\mu \sigma^\mu \,. \tag{3.1}$$

Since the hermitean matrices over the field of complex numbers close under the symmetric anti-commutator product one can regard the coordinate vectors as elements of a Jordan algebra denoted as $J_2^{\mathbb{C}}$ [1, 2]. Then the rotation, Lorentz and conformal groups in $d = 4$ can be identified with the automorphism, reduced structure and Möbius (linear fractional) groups of the Jordan algebra of 2×2 complex hermitean matrices $J_2^{\mathbb{C}}$ [1, 2]. The reduced structure group $Str_0(J)$ of a Jordan algebra J is simply the invariance group of its norm form $N(J)$. (The structure group $Str(J) = Str_0(J) \times SO(1,1)$, on the other hand, is simply the invariance group of $N(J)$ up to an overall constant scale factor.) Furthermore, this interpretation leads one naturally to define generalized space-times whose coordinates are parameterized by the elements of Jordan algebras [1]. The rotation $Rot(J)$, Lorentz $Lor(J)$ and conformal $Con(J)$ groups of these generalized space-times are then identified with the automorphism $Aut(J)$, reduced structure $Str_0(J)$ and Möbius Mö(J) groups of the corresponding Jordan algebras [1–4]. Denoting as $J_n^{\mathbb{A}}$ the Jordan algebra of $n \times n$ hermitean matrices over the *division* algebra $\mathbb{A}$ and the Jordan algebra of Dirac gamma matrices in d (euclidean) dimensions as $\Gamma(d)$ one finds the following symmetry groups of generalized space-times defined by simple euclidean (formally real) Jordan algebras:

J	Rotation(J)	Lorentz(J)	Conformal(J)
$J_n^{\mathbb{R}}$	$\mathrm{SO}(n)$	$\mathrm{SL}(n,\mathbb{R})$	$\mathrm{Sp}(2n,\mathbb{R})$
$J_n^{\mathbb{C}}$	$\mathrm{SU}(n)$	$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{SU}(n,n)$
$J_n^{\mathbb{H}}$	$\mathrm{USp}(2n)$	$\mathrm{SU}^*(2n)$	$SO^*(4n)$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$
$\Gamma(d)$	$\mathrm{SO}(d)$	$\mathrm{SO}(d,1)$	$\mathrm{SO}(d,2)$

The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ represent the four division algebras. For the Jordan algebras $J_n^{\mathbb{A}}$ the norm form is the determinental form (or its generalization to the quaternionic and octonionic matrices). For the Jordan algebra $\Gamma(d)$ generated by Dirac gamma matrices Γ_i ($i = 1, 2, \ldots, d$)

$$\{\Gamma_i, \Gamma_j\} = \delta_{ij} \mathbf{1} \,; \qquad i, j, \ldots = 1, 2, \ldots, d \tag{3.2}$$

the norm of a general element $x = x_0\mathbf{1} + x_i\Gamma_i$ of $\Gamma(d)$ is quadratic and given by

$$N(x) = x\bar{x} = x_0^2 - x_i x_i \tag{3.3}$$

where $\bar{x} = x_0\mathbf{1} - x_i\Gamma_i$. Its automorphism, reduced structure and Möbius groups are simply the rotation, Lorentz and conformal groups of $(d+1)$-dimensional Minkowski spacetime. One finds the following special isomorphisms between the Jordan algebras of 2×2 hermitean matrices over the four division algebras and the Jordan algebras of gamma matrices:

$$J_2^{\mathbb{R}} \simeq \Gamma(2)\,; \qquad J_2^{\mathbb{C}} \simeq \Gamma(3)\,; \qquad J_2^{\mathbb{H}} \simeq \Gamma(5)\,; \qquad J_2^{\mathbb{O}} \simeq \Gamma(9)\,. \tag{3.4}$$

The Minkowski spacetimes they correspond to are precisely the critical dimensions for the existence of super Yang-Mills theories as well as of the classical Green-Schwarz superstrings. These Jordan algebras are all quadratic and their norm forms are precisely the quadratic invariants constructed using the Minkowski metric.

We should note two remarkable facts about the above table. First, the conformal groups of generalized space-times defined by euclidean (formally real) Jordan algebras all admit positive energy unitary representations.[4] Hence they can be given a causal structure with a unitary time evolution as in four dimensional Minkowski space-time. Second is the fact that the maximal compact subgroups of the generalized conformal groups of formally real Jordan algebras are simply the compact forms of their structure groups (which are the products of their generalized Lorentz groups with dilatations).

3.2 Quasiconformal groups and Freudenthal triple systems

A Freudenthal triple system (FTS) is a vector space $\mathfrak{M}$ with a trilinear product (X,Y,Z) and a skew symmetric bilinear form $\langle X,Y\rangle$ such that:[5]

$$\begin{aligned}
(X,Y,Z) &= (Y,X,Z) + 2\,\langle X,Y\rangle\, Z\,,\\
(X,Y,Z) &= (Z,Y,X) - 2\,\langle X,Z\rangle\, Y\,,\\
\langle (X,Y,Z)\,, W\rangle &= \langle (X,W,Z)\,, Y\rangle - 2\,\langle X,Z\rangle\,\langle Y,W\rangle\,,\\
(X,Y,(V,W,Z)) &= (V,W,(X,Y,Z)) + ((X,Y,V)\,,W,Z) +\\
&\quad + (V,(Y,X,W)\,,Z)\;.
\end{aligned} \tag{3.5}$$

A quartic invariant $\mathcal{I}_4$ can be constructed over the FTS by means of the triple product and the bilinear form as

$$\mathcal{I}_4(X) := \langle (X,X,X)\,, X\rangle\;. \tag{3.6}$$

[4]Similarly, the generalized conformal groups defined by hermitean Jordan triple systems all admit positive energy unitary representations [4]. In fact the conformal groups of simple hermitean Jordan triple systems exhaust the list of simple noncompact groups that admit positive energy unitary representations. They include the conformal groups of simple euclidean Jordan algebra since the latter form an hermitian Jordan triple system under the Jordan triple product [4].

[5]We should note that the triple product (3.5) could be modified by terms involving the symplectic invariant, such as $\langle X,Y\rangle Z$. The choice given above was made in [9] in order to obtain agreement with the formulas of [22].

One can construct a Lie algebra over a FTS that has a 5-graded decomposition such that grade ± 2 subspaces are one dimensional:

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} . \tag{3.7}$$

Following [9] we shall label the Lie algebra generators belonging to grade $+1$ and grade -1 subspaces as U_A and $\tilde{U}_A$, where $A \in \mathfrak{M}$. The generators S_{AB} belonging to grade zero subspace are labeled by a pair of elements $A, B \in \mathfrak{M}$. For the grade ± 2 subspaces one would in general need another set of generators K_{AB} and $\tilde{K}_{AB}$ labeled by two elements, but since these subspaces are one-dimensional we can write them as

$$K_{AB} = \langle A, B \rangle \, K_a \qquad \tilde{K}_{AB} = \langle A, B \rangle \, \tilde{K}_a \tag{3.8}$$

where a is a real parameter.

One can realize the Lie algebra $\mathfrak{g}$ as a quasiconformal Lie algebra over a vector space whose coordinates $\mathcal{X}$ are labeled by a pair (X, x), where $X \in \mathfrak{M}$ and x is an extra single variable as follows [9]:

$$\begin{aligned}
K_a\,(X) &= 0 \qquad U_A\,(X) = A \qquad S_{AB}\,(X) = (A, B, X) \\
K_a\,(x) &= 2\,a \qquad U_A\,(x) = \langle A, X \rangle \qquad S_{AB}\,(x) = 2\,\langle A, B \rangle\, x \\
\tilde{U}_A\,(X) &= \frac{1}{2}\,(X, A, X) - Ax \\
\tilde{U}_A\,(x) &= -\frac{1}{6}\,\langle (X, X, X)\,, A \rangle + \langle X, A \rangle\, x \\
\tilde{K}_a\,(X) &= -\frac{1}{6}\,a\,(X, X, X) + aXx \\
\tilde{K}_a\,(x) &= \frac{1}{6}\,a\,\langle (X, X, X)\,, X \rangle + 2\,a\,x^2 .
\end{aligned} \tag{3.9}$$

From these formulas it is straightforward to determine the commutation relations of the transformations. The quasiconformal groups leave invariant a suitable defined lightcone with respect to a quartic norm involving the quartic invariant of $\mathfrak{M}$ [9].

Freudenthal introduced the triple systems associated with his name in his study of the metasymplectic geometries associated with exceptional groups [25]. The geometries associated with FTSs were further studied in [24, 22, 15, 16]. A classification of FTS's may be found in [16], where it is also shown that there is a one-to-one correspondence between simple Lie algebras and simple FTS's with a non-degenerate skew symmetric bilinear form. Hence there is a quasiconformal realization of every Lie group acting on a generalized lightcone.

The Freudenthal triple systems associated with exceptional groups can be represented by formal 2×2 "matrices" of the form

$$A = \begin{pmatrix} \alpha_1 & x_1 \\ x_2 & \alpha_2 \end{pmatrix} , \tag{3.10}$$

where α_1, α_2 are real numbers and x_1, x_2 are elements of a simple Jordan algebra J of degree three. One can define a triple product over the space of such formal matrices such that they close under it. There are only four simple euclidean Jordan algebras J of this type, namely the 3×3 hermitean matrices over the four division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. We shall denote the corresponding FTS's as $\mathfrak{M}(J)$.

One may ask which Freudenthal triple systems can be realized in the above form in terms of an underlying Jordan algebra. This question was investigated by Ferrar [19] who proved that such a realization is possible only if the underlying Jordan algebra is of degree three. Remarkably, if one further requires that the underlying Jordan algebra be formally real then the list of Jordan algebras over which FTS's can be defined as above coincides with the list of Jordan algebras that occur in five dimensional $N = 2$ MESGT's whose target spaces are symmetric spaces of the form G/H such that G is a symmetry of the lagrangian.

In this paper we will focus only on the quasiconformal groups defined over formally real Jordan algebras. The Freudenthal triple product of the elements of $\mathfrak{M}(J)$ is defined as [22]

$$\langle X_1, X_2, X_3 \rangle = \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} \qquad \text{with} \qquad X_i = \begin{pmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{pmatrix} \tag{3.11}$$

where

$$\begin{aligned} \gamma &= \alpha_1\beta_2\alpha_3 + 2\alpha_1\alpha_2\beta_3 - \alpha_3 T(a_1, b_2) - \alpha_2 T(a_1, b_3) - \\ &\quad -\alpha_1 T(a_2, b_3) + T(a_1, a_2 \times a_3) \\ c &= (\alpha_2\beta_3 + T(b_2, a_3))\, a_1 + (\alpha_1\beta_3 + T(b_1, a_3))\, a_2 + (\alpha_1\beta_2 + T(b_1, a_2))\, a_3 - \\ &\quad -\alpha_1\, b_2 \times b_3 - \alpha_2\, b_1 \times b_3 - \alpha_3 b_1 \times b_2 - \\ &\quad - \{a_1, b_2, a_3\} - \{a_1, b_3, a_2\} - \{a_2, b_1, a_3\} \\ \delta &= -\gamma^\sigma \qquad d = -c^\sigma \qquad \text{where} \qquad \sigma = (\alpha \leftrightarrow \beta)\,(a \leftrightarrow b)\,. \end{aligned} \tag{3.12}$$

Here σ denotes a permutation of α with β and a with b, and

$$\{a, b, c\} = U_{a+c} b - U_a b - U_c b \tag{3.13}$$

where $U_a b$ is defined as in (2.14).

4. Spacetimes over Jordan algebras of degree three as dilatonic and spinorial extensions of Minkowskian spacetimes

As stated above we will restrict our studies of generalized spacetimes to those defined by formally real Jordan algebras of degree 3. Our main goal is to give a unified geometric realization of the conformal and quasiconformal groups of generalized spacetimes defined by Jordan algebras of degree three and the FTS's defined over them.

The related geometries in the context of $N = 2$ MESGT's were reviewed in section 2.2. The Jordan algebras of degree three that arose in the study of MESGT's were later studied by Sierra who showed that there exists a correspondence between them and classical relativistic point particle actions [18]. In the same work Sierra showed that this could be extended to a correspondence between classical relativistic bosonic strings and the Freudenthal triple systems defined over them.

Consider now the spacetimes coordinatized by the generic Jordan family

$$J = \mathbb{R} \oplus \Gamma(Q) \tag{4.1}$$

we shall interpret the extra coordinate corresponding to $\mathbb{R}$ as a dilatonic coordinate ρ and label the coordinates defined by J as $(\rho, x_m, m = 0, 1, 2, \dots (d-1))$. The automorphism group $\mathrm{SO}(d-1)$ will then be the rotation group of this space-time under which both the time coordinate x_0 and the dilatonic coordinate ρ will be singlets. The Lorentz group of this spacetime is the reduced structure group which is simply

$$\mathrm{SO}(d-1,1) \times \mathrm{SO}(1,1)\,. \tag{4.2}$$

It leaves invariant the cubic norm which, following [18], we normalize as

$$\mathcal{V}\left(\rho, x_m\right) = \sqrt{2}\rho\, x_m x_n \eta^{mn}\,. \tag{4.3}$$

Under the action of $\mathrm{SO}(d-1,1)$, the dilaton ρ is a singlet and under $\mathrm{SO}(1,1)$ we have

$$SO(1,1): \quad \begin{array}{c} \rho \Rightarrow e^{2\lambda}\rho \\ x_m \Rightarrow e^{\lambda}x_m \end{array}\,. \tag{4.4}$$

Freudenthal product of two elements of $J = \mathbb{R} \oplus \Gamma(Q)$ is simply

$$(\rho, x) \times (\sigma, y) = \left(\sqrt{2}x_m y^m, \sqrt{2}\left(\rho y_m + \sigma x_m\right).\right) \tag{4.5}$$

The conformal group of the spacetime is the Möbius group of J which is

$$\mathrm{SO}\,(d,2) \times \mathrm{SO}\,(2,1)\,. \tag{4.6}$$

The Freudenthal triple systems defined over the generic Jordan family can be represented by 2×2 matrices

$$\mathfrak{M}\left(J = \mathbb{R} \oplus \Gamma\left(Q\right)\right) = \begin{pmatrix} x_d^1 & J^1 \\ J^2 & x_d^2 \end{pmatrix} = X \tag{4.7}$$

where $J^1, J^2 \in J$ and x_d^1 and x_d^2 are real coordinates. The automorphism group of $\mathfrak{M}$ is $\mathrm{SO}\,(d,2) \otimes \mathrm{Sp}\,(2,\mathbb{R})$ under which an element of $\mathfrak{M}$ transforms in the representation $(\mathbf{d+2}, \mathbf{2})$. We shall label the "coordinates" of $\mathfrak{M}$ as

$$x_\mu^a = \left(x_m^a, x_d^a, \rho^a\right) \qquad \text{where} \qquad a = 1, 2$$

and interpret it as coordinates of a conformally covariant phase space (so that $a = 1$ labels the coordinates and $a = 2$ labels the momenta).

Skew-symmetric invariant form over $\mathfrak{M}$ is given by

$$\langle X, Y \rangle = \epsilon_{ab}\eta^{\mu\nu} X_\mu^a Y_\nu^b\,. \tag{4.8}$$

We should stress the important fact that the conformal group of the spacetime defined by J is isomorphic to the automorphism group of the Freudenthal triple system $\mathfrak{M}\,(J)$!

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To define the quasi-conformal group over the conformal phase space represented by $\mathfrak{M}(J)$ we need to extend it by an extra coordinate corresponding to the cocycle (symplectic form) over $\mathfrak{M}(J)$. We shall denote the elements of $\mathfrak{M}(J)$ as X and the extra coordinate as x. The quasi-conformal group of $\mathfrak{M}(J) \oplus \mathbb{R}$ is the group $\mathrm{SO}(d+2,4)$.

The space-times defined by simple Jordan algebras of degree 3 $J_3^{\mathbb{A}}$ correspond to extensions of Minkowski space-times in the critical dimensions $d = 3,4,6,10$ by a dilatonic (ρ) *and* commuting spinorial coordinates (ξ^a).

$$J_3^{\mathbb{R}} \Longleftrightarrow (\rho, x_m, \xi^\alpha) \qquad m = 0,1,2 \qquad \alpha = 1,2 \tag{4.9}$$
$$J_3^{\mathbb{C}} \Longleftrightarrow (\rho, x_m, \xi^\alpha) \qquad m = 0,1,2,3 \quad \alpha = 1,2,3,4 \tag{4.10}$$
$$J_3^{\mathbb{H}} \Longleftrightarrow (\rho, x_m, \xi^\alpha) \qquad m = 0,\dots,5 \quad \alpha = 1,\dots,8 \tag{4.11}$$
$$J_3^{\mathbb{C}} \Longleftrightarrow (\rho, x_m, \xi^\alpha) \qquad m = 0,\dots,9 \quad \alpha = 1,\dots,16\,. \tag{4.12}$$

The commuting spinors ξ are represented by a 2×1 matrix over $\mathbb{A} = \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$. The cubic norm of a "vector" with coordinates (ρ, x_m, ξ^α) is given by [18]

$$\mathcal{V}(\rho, x_m, \xi^\alpha) = \sqrt{2}\rho x_m x_n \eta^{mn} + x^m \bar{\xi}\gamma_m \xi\,. \tag{4.13}$$

The Lorentz groups of the space-times over $J_3^{\mathbb{A}}$ are

$$\mathrm{SL}(3,\mathbb{R})\,, \qquad \mathrm{SL}(3,\mathbb{C})\,, \qquad \mathrm{SU}^*(6)\,, \qquad \text{and} \qquad \mathrm{E}_{6(-26)} \tag{4.14}$$

respectively, corresponding to the invariance groups of their cubic norm. The Freudenthal product of two vectors in the corresponding space-time is given by

$$\begin{aligned}&(\rho, x_m, \xi^\alpha) \times (\sigma, y_m, \zeta^\alpha) = \\ &= \left(\sqrt{2}\,x_m y^m,\ \frac{1}{2}\left(\bar{\xi}\gamma_m\zeta + \bar{\zeta}\gamma_m\xi\right) + \sqrt{2}\left(\rho\, y_m + \sigma\, x_m\right),\ x^m\,\bar{\zeta}\gamma_m + y^m\,\bar{\xi}\gamma_m\right)\end{aligned} \tag{4.15}$$

The conformal groups of these spacetimes are

$$\mathrm{Sp}(6,\mathbb{R})\,, \qquad \mathrm{SU}(3,3)\,, \qquad \mathrm{SO}^*(12)\,, \qquad \text{and} \qquad \mathrm{E}_{7(-25)} \tag{4.16}$$

respectively. The automorphism groups of the FTS $\mathfrak{M}\left(J_3^{\mathbb{A}}\right)$ are isomorphic to their conformal groups.

The quasi-conformal groups acting on $\mathfrak{M}\left(J_3^{\mathbb{A}} \oplus \mathbb{R}\right)$, where $\mathbb{R}$ represents the extra "cocyle" coordinate, are

$$\mathrm{F}_{4(4)}\,, \qquad \mathrm{E}_{6(2)}\,, \qquad \mathrm{E}_{7(-5)}\,, \qquad \mathrm{E}_{8(-24)} \tag{4.17}$$

whose minimal unitary irreducible representations were constructed in [10].

5. Geometric realizations of $\mathrm{SO}(d+2,4)$ as quasiconformal groups

Lie algebra of $\mathrm{SO}(d+2,4)$ admits the following 5-graded decomposition

$$\mathfrak{so}(d+2,4) = \mathbf{1} \oplus (\mathbf{d+2,2}) \oplus (\Delta \oplus \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{so}(d,2)) \oplus (\mathbf{d+2,2}) \oplus \mathbf{1}\,. \tag{5.1}$$

Generators are realized as differential operators in $2d+5$ coordinates corresponding to $\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$ subspace which we shall denote as x and $X^{\mu,a}$ where $a = 1,2$ is an index of representation $\mathbf{2}$ of $\mathfrak{sp}(2,\mathbb{R})$ and we shall let the indices μ run from 1 to $d+2$ with the indices $d+1$ and $d+2$ labelling the two timelike coordinates, i.e x_μ transforms like a vector of $\mathrm{SO}(d,2)$.

Let ϵ_{ab} be symplectic real-valued matrix, and $\eta_{\mu\nu}$ denote signature $(d,2)$ metric preserved by $\mathrm{SO}(d,2)$. Then

$$\mathcal{I}_4 = \eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd}X^{\mu,a}X^{\nu,b}X^{\rho,c}X^{\tau,d} \tag{5.2}$$

is a 4th-order polynomial invariant under the semisimple part of $\mathfrak{g}^0$. Define

$$\begin{aligned}
K_+ &= \frac{1}{2}\left(2x^2 - \mathcal{I}_4\right)\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} + x\, X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} \\
U_{\mu,a} &= \frac{\partial}{\partial X^{\mu,a}} - \eta_{\mu,\nu}\epsilon_{ab}X^{\nu,b}\frac{\partial}{\partial x} \\
M_{\mu\nu} &= \eta_{\mu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\mu,a}} \\
J_{ab} &= \epsilon_{ac}X^{\mu,c}\frac{\partial}{\partial X^{\mu,b}} + \epsilon_{bc}X^{\mu,c}\frac{\partial}{\partial X^{\mu,a}} \\
K_- &= \frac{\partial}{\partial x} \qquad \Delta = 2x\frac{\partial}{\partial x} + X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} \qquad \tilde{U}_{\mu,a} = [U_{\mu,a}, K_+]
\end{aligned} \tag{5.3}$$

where ϵ^{ab} denotes an inverse symplectic metric: $\epsilon^{ab}\epsilon_{bc} = \delta^a{}_c$ and $\tilde{U}_{\mu,a}$ evaluates to

$$\begin{aligned}
\tilde{U}_{\mu,a} = \eta_{\mu\nu}\epsilon_{ad}&\left(\eta_{\lambda\rho}\epsilon_{bc}X^{\nu,b}X^{\lambda,c}X^{\rho,d} - xX^{\nu,d}\right)\frac{\partial}{\partial x} + x\frac{\partial}{\partial X^{\mu,a}} - \\
&-\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}X^{\rho,c}\frac{\partial}{\partial X^{\rho,c}} - \epsilon_{ad}\eta_{\lambda\rho}X^{\rho,d}X^{\lambda,c}\frac{\partial}{\partial X^{\mu,c}} + \\
&+\epsilon_{ad}\eta_{\mu\nu}X^{\rho,d}X^{\nu,b}\frac{\partial}{\partial X^{\rho,b}} + \eta_{\mu\nu}\epsilon_{bc}X^{\nu,b}X^{\rho,c}\frac{\partial}{\partial X^{\rho,a}}
\end{aligned} \tag{5.4}$$

we have

$$\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}} = -4\,\eta_{\mu\nu}\,\eta_{\lambda\rho}\,X^{\nu,b}X^{\lambda,c}X^{\rho,d}\,\epsilon_{bc}\epsilon_{ad}\,.$$

These generators satisfy the following commutation relation:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\tau}] &= \eta_{\nu\rho}M_{\mu\tau} - \eta_{\mu\rho}M_{\nu\tau} + \eta_{\mu\tau}M_{\nu\rho} - \eta_{\nu\tau}M_{\mu\rho} \\
[J_{ab}, J_{cd}] &= \epsilon_{cb}J_{ad} + \epsilon_{ca}J_{bd} + \epsilon_{db}J_{ac} + \epsilon_{da}J_{bc} \\
[\Delta, K_\pm] &= \pm 2K_\pm \qquad [K_-, K_+] = \Delta \\
[\Delta, U_{\mu,a}] &= -U_{\mu,a} \qquad \left[\Delta, \tilde{U}_{\mu,a}\right] = \tilde{U}_{\mu,a} \\
[U_{\mu,a}, K_+] &= \tilde{U}_{\mu,a} \qquad \left[\tilde{U}_{\mu,a}, K_-\right] = -U_{\mu,a} \\
[U_{\mu,a}, U_{\nu,b}] &= 2\eta_{\mu\nu}\epsilon_{ab}K_- \qquad \left[\tilde{U}_{\mu,a}, \tilde{U}_{\nu,b}\right] = 2\eta_{\mu\nu}\epsilon_{ab}K_+
\end{aligned} \tag{5.5a}$$

$$\begin{aligned}
[M_{\mu\nu}, U_{\rho,a}] &= \eta_{\nu\rho}U_{\mu,a} - \eta_{\mu\rho}U_{\nu,a} \qquad \left[M_{\mu\nu}, \tilde{U}_{\rho,a}\right] = \eta_{\nu\rho}\tilde{U}_{\mu,a} - \eta_{\mu\rho}\tilde{U}_{\nu,a} \\
[J_{ab}, U_{\mu,c}] &= \epsilon_{cb}U_{\mu,a} + \epsilon_{ca}U_{\mu,b} \qquad \left[J_{ab}, \tilde{U}_{\mu,c}\right] = \epsilon_{cb}\tilde{U}_{\mu,a} + \epsilon_{ca}\tilde{U}_{\mu,b}
\end{aligned} \tag{5.5b}$$

$$\left[U_{\mu,a},\tilde{U}_{\nu,b}\right] = \eta_{\mu\nu}\epsilon_{ab}\,\Delta - 2\,\epsilon_{ab}M_{\mu\nu} - \eta_{\mu\nu}J_{ab}\,. \tag{5.5c}$$

The distance invariant under $\mathrm{SO}\,(d+2,4)$ can be constructed following [9]. Let us first introduce a difference between two vectors $\mathcal{X}$ and $\mathcal{Y}$ on $\mathfrak{g}^{-1}\oplus\mathfrak{g}^{-2}$:

$$\delta\left(\mathcal{X},\mathcal{Y}\right) = \left(X^{\mu,a} - Y^{\mu,a}, x - y - \eta_{\mu\nu}\epsilon_{ab}X^{\mu,a}Y^{\nu,b}\right) \tag{5.6}$$

and define the "length" of a vector $\mathcal{X}$ as

$$\ell\left(\mathcal{X}\right) = \mathcal{I}_4\left(X\right) + 2x^2\,. \tag{5.7}$$

Then the cone defined by $\ell\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right) = 0$ is invariant w.r.t. the full group $\mathrm{SO}(d+2,4)$, because of the following identities:

$$\begin{aligned}
\Delta\circ\ell\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right) &= 4\,\ell\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right)\\
\tilde{U}_{\mu,a}\circ\ell\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right) &= -2\eta_{\mu\nu}\epsilon_{ab}\left(X^{\nu,b}+Y^{\nu,b}\right)\ell\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right)\\
K_+\circ\ell\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right) &= 2\left(x+y\right)\ell\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right)\\
\text{any other generator}\circ\ell\left(\delta\left(\mathcal{X},\mathcal{Y}\right)\right) &= 0
\end{aligned} \tag{5.8}$$

6. Geometric realizations of $\mathrm{E}_{8(-24)}$, $\mathrm{E}_{7(-5)}$, $\mathrm{E}_{6(2)}$ and $\mathrm{F}_{4(4)}$ as quasiconformal groups

The minimal unitary representations of the quasiconformal groups of the spacetimes defined by simple formally real Jordan algebras of degree three were given in our earlier paper [10]. In this section we will give their geometric realizations as quasiconformal groups in an $\mathrm{SO}(d,2)\times Sp(2,\mathbb{R})$ covariant basis where d is equal to one of the critical dimensions $3,4,6,10$.

6.1 Geometric realization of the quasiconformal group $\mathrm{E}_{8(-24)}$

For realizing the geometric action of the quasiconformal group $\mathrm{E}_{8(-24)}$ in an $\mathrm{SO}(10,2)\times\mathrm{SO}(2,1)$ covariant basis we shall use the following 5-graded decomposition of its Lie algebra

$$\mathfrak{e}_{8(-24)} = \tilde{\mathbf{1}}\oplus\tilde{\mathbf{56}}\oplus\left[\mathfrak{so}(1,1)\oplus\mathfrak{e}_{7(-25)}\right]\oplus\mathbf{56}\oplus\mathbf{1} \tag{6.1}$$

$$\mathfrak{e}_{8(-24)} = \mathbf{1}\oplus\begin{pmatrix}(\mathbf{2},\mathbf{12})\\(\mathbf{1},\mathbf{32}_c)\end{pmatrix}\oplus \tag{6.2}$$

$$\oplus\left[\Delta\oplus\begin{pmatrix}(\mathbf{2},\mathbf{32}_s)\\\mathfrak{sp}\left(2,\mathbb{R}\right)\oplus\mathfrak{so}\left(10,2\right)\end{pmatrix}\right]\oplus\begin{pmatrix}(\mathbf{2},\mathbf{12})\\(\mathbf{1},\mathbf{32}_c)\end{pmatrix}\oplus\mathbf{1}\,. \tag{6.3}$$

The generators of the simple subalgebra $\mathfrak{e}_{7(-25)}$ in $\mathfrak{g}^0$ satisfy the following $\mathrm{SO}(10,2)$ covariant commutation relations.

$$\begin{aligned}
\left[M_{\mu\nu},Q_{a\dot{\alpha}}\right] &= Q_{a\dot{\beta}}\left(\Gamma_{\mu\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}}\\
\left[J_{ab},Q_{c\dot{\alpha}}\right] &= \epsilon_{cb}Q_{a\dot{\alpha}} + \epsilon_{ca}Q_{b\dot{\alpha}}
\end{aligned}$$

$$\left[Q_{a\dot{\alpha}}, Q_{b\dot{\beta}}\right] = \epsilon_{ab} \left(C\Gamma_{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}} M^{\mu\nu} + C_{\dot{\alpha}\dot{\beta}} J_{ab} \tag{6.4}$$

where $M_{\mu\nu}$ and J_{ab} are the generators of SO$(10,2)$ and Sp$(2,\mathbb{R})$, respectively and $Q_{a\dot{\alpha}}$ are the remaining generators transforming in the $(\mathbf{32}_s, \mathbf{2}))$ of SO$(10,2) \times Sp(2,\mathbb{R})$. C is the charge conjugation matrix in $(10,2)$ dimensions and is antisymmetric

$$C^t = -C\,. \tag{6.5}$$

The generators of $\mathrm{E}_{7(-25)}$ are realized in terms of the "coordinates" $X^{\mu,a}$ and ψ^α transforming in the $(\mathbf{12},\mathbf{2})$ and $(\mathbf{32}_c,\mathbf{1})$ representation of SO$(10,2) \times Sp(2,\mathbb{R})$ as follows

$$\begin{aligned}
M_{\mu\nu} &= \eta_{\mu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\mu,a}} - \psi^\alpha (\Gamma_{\mu\nu})^\beta{}_\alpha \frac{\partial}{\partial \psi^\beta} \\
J_{ab} &= \epsilon_{ac} X^{\mu,c} \frac{\partial}{\partial X^{\mu,b}} + \epsilon_{bc} X^{\mu,c} \frac{\partial}{\partial X^{\mu,a}} \\
Q_{a\dot{\alpha}} &= \epsilon_{ab} X^{\mu,b} (\Gamma_\mu)^\beta{}_{\dot{\alpha}} \frac{\partial}{\partial \psi^\beta} - \psi^\beta \left(C\Gamma_\mu\right)_{\beta\dot{\alpha}} \eta^{\mu\nu} \frac{\partial}{\partial X^{\nu,a}}
\end{aligned} \tag{6.6}$$

where Γ_μ are the gamma matrices and $\Gamma_{\mu\nu} = \frac{1}{4}\left(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu\right)$. $\alpha, \beta, \ldots$ and $\dot{\alpha}, \dot{\beta}, \ldots$ are chiral and anti-chiral spinor indices that run from 1 to 32, respectively. Γ matrices are taken to be in a chiral basis (with Γ_{13} being diagonal). The spinorial "coordinates" ψ^α transform as a Majorana-Weyl spinor of SO$(10,2)$. One convenient choice for gamma matrices is

$$\begin{aligned}
&\Gamma_i = \sigma_1 \otimes \sigma_1 \otimes \Gamma_i^{(8)} \qquad \Gamma_9 = \sigma_1 \otimes \sigma_1 \otimes \Gamma_9^{(8)} \qquad \Gamma_{10} = \sigma_1 \otimes \sigma_3 \otimes 1_{16} \\
&\Gamma_{11} = \sigma_1 \otimes i\sigma_2 \otimes 1_{16} \qquad \Gamma_{12} = i\sigma_2 \otimes 1_{32} \qquad C = 1_2 \otimes i\sigma_2 \otimes 1_{16}
\end{aligned} \tag{6.7}$$

where $i = 1, \ldots, 8$ and $\Gamma_9^{(8)} = \Gamma_1^{(8)} \cdots \Gamma_8^{(8)}$. Matrices $\Gamma_i^{(8)}$ are those of Clifford algebra of $\mathbb{R}^8$. The chiral realization (6.7) assumes mostly plus signature convention:

$$\eta_{\mu\nu} = \mathrm{diag}\left((+)^{10}, (-)^2\right) \qquad \mu,\nu = 1, \ldots, 12\,. \tag{6.8}$$

The fourth order invariant of $\mathfrak{e}_{7(-25)}$ in the above basis reads as

$$\begin{aligned}
\mathcal{I}_4 &= \eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd} X^{\mu,a} X^{\nu,b} X^{\rho,c} X^{\tau,d} + 2\epsilon_{ab} X^{\mu,a} X^{\nu,b} \psi^\alpha \left(C\Gamma_{\mu\nu}\right)_{\alpha\beta} \psi^\beta + \\
&+ \frac{1}{6} \psi^\alpha \left(C\Gamma_{\mu\nu}\right)_{\alpha\beta} \psi^\beta \psi^\gamma \left(C\Gamma^{\mu\nu}\right)_{\gamma\delta} \psi^\delta\,.
\end{aligned} \tag{6.9}$$

Given the above data, it is straightforward to realize the generators of $\mathfrak{e}_{8(-24)}$ on a $56+1 = 57$ dimensional space following [9, 28, 10]. We start with negative grade generators

$$\begin{aligned}
K_- &= \frac{\partial}{\partial x} \qquad U_\alpha = \frac{\partial}{\partial \psi^\alpha} - C_{\alpha\beta}\psi^\beta \frac{\partial}{\partial x} \\
U_{\mu,a} &= \frac{\partial}{\partial X^{\mu,a}} - \eta_{\mu,\nu}\epsilon_{ab} X^{\nu,b} \frac{\partial}{\partial x}
\end{aligned} \tag{6.10}$$

where x is the singlet "cocycle" coordinate. Grade $+2$ generator is

$$K_+ = \frac{1}{2}\left(2x^2 - \mathcal{I}_4\right) \frac{\partial}{\partial x} - \frac{1}{4} \frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}} \eta^{\mu\nu} \epsilon^{ab} \frac{\partial}{\partial X^{\nu,b}} +$$

$$+\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial \psi^\alpha}\left(C^{-1}\right)^{\alpha\beta}\frac{\partial}{\partial \psi^\beta}+x\,X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}}+x\,\psi^\alpha\frac{\partial}{\partial \psi^\alpha}\,. \tag{6.11}$$

Generators of grade +1 space are obtained by commuting K_+ with corresponding generators of $\mathfrak{g}^{-1}$:

$$\tilde{U}_{\mu,a}=[U_{\mu,a},K_+] \qquad \tilde{U}_\alpha=[U_\alpha,K_+]\,. \tag{6.12}$$

The generator that determines the five grading is simply

$$\Delta=2x\frac{\partial}{\partial x}+X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}}+\psi^\alpha\frac{\partial}{\partial \psi^\alpha}\,. \tag{6.13}$$

The commutation relations of these generators are those of (5.5) for $d=10$ supplemented with (6.4) and the following:

$$\begin{aligned}
[U_\alpha,U_\beta] &= 2\,C_{\alpha\beta}K_- \qquad [U_\alpha,K_+]=\tilde{U}_\alpha\\
\left[\tilde{U}_\alpha,\tilde{U}_\beta\right] &= 2\,C_{\alpha\beta}K_+ \qquad \left[\tilde{U}_\alpha,K_-\right]=-U_\alpha\\
[Q_{a\dot{\alpha}},U_{\mu,b}] &= -\epsilon_{ab}(\Gamma_\mu)^\alpha{}_{\dot{\alpha}}U_\alpha \qquad [Q_{a\dot{\alpha}},U_\beta]=(C\Gamma_\mu)_{\beta\dot{\alpha}}\,\eta^{\mu\nu}U_{\nu,a}\\
\left[Q_{a\dot{\alpha}},\tilde{U}_{\mu,b}\right] &= -\epsilon_{ab}(\Gamma_\mu)^\alpha{}_{\dot{\alpha}}\tilde{U}_\alpha \qquad \left[Q_{a\dot{\alpha}},\tilde{U}_\beta\right]=(C\Gamma_\mu)_{\beta\dot{\alpha}}\,\eta^{\mu\nu}\tilde{U}_{\nu,a}\\
\left[\tilde{U}_\alpha,U_{\mu,a}\right] &= -\left(C\Gamma_\mu C^{-1}\right)_\alpha{}^{\dot{\alpha}}Q_{a\dot{\alpha}} \qquad \left[U_\alpha,\tilde{U}_\beta\right]=C_{\alpha\beta}\Delta-(C\Gamma_{\mu\nu})_{\alpha\beta}\,M^{\mu\nu}\\
\left[U_\alpha,\tilde{U}_{\mu,a}\right] &= \left(C\Gamma_\mu C^{-1}\right)_\alpha{}^{\dot{\alpha}}Q_{a\dot{\alpha}}
\end{aligned} \tag{6.14}$$

with all the remaining commutators vanishing. The explicit expressions for the grade +1 generators are

$$\begin{aligned}
\tilde{U}_{\mu,a} &= -\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\frac{\partial}{\partial x}-x\,\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\frac{\partial}{\partial x}+x\frac{\partial}{\partial X^{\mu,a}}-\\
&\quad -\frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\partial X^{\nu,b}}\eta^{\nu\lambda}\epsilon^{bc}\frac{\partial}{\partial X^{\lambda,c}}-\frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\partial \psi^\alpha}\left(C^{-1}\right)^{\alpha\beta}\frac{\partial}{\partial \psi^\beta}-\\
&\quad -\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}}+\psi^\gamma\frac{\partial}{\partial \psi^\gamma}\right) &(6.15)\\
\tilde{U}_\alpha &= -\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial \psi^\alpha}\frac{\partial}{\partial x}-x\left(C_{\alpha\beta}\psi^\beta\right)\frac{\partial}{\partial x}-C_{\alpha\beta}\psi^\beta\left(X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}}+\psi^\gamma\frac{\partial}{\partial \psi^\gamma}\right)-\\
&\quad -\frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\partial \psi^\alpha}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}}-\frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial \psi^\alpha\partial \psi^\beta}\left(C^{-1}\right)^{\beta\gamma}\frac{\partial}{\partial \psi^\gamma}+x\frac{\partial}{\partial \psi^\alpha}\,. &(6.16)
\end{aligned}$$

The above geometric realization of the quasiconformal action of the Lie algebra of $E_{8(-24)}$ can be consistently truncated to the quasiconformal realizations of $E_{7(-5)}$, $E_{6(2)}$ and $F_{4(4)}$, which we discuss in the following subsections. We should stress that for all these groups one can define a quartic norm such that they leave the generalized light-cone defined with respect to this quartic norm invariant as was shown for the maximally split exceptional groups in [9] and for $SO(d+2,4)$ in section 5 above.

6.2 Geometric realization of the quasiconformal group $E_{7(-5)}$

Truncation of the geometric realization of the quasiconformal group $E_{8(-24)}$ to $\mathfrak{e}_{7(-5)}$ is achieved by "dimensional reduction" from 10 to 6 dimensions. Reduction of 32-component

Majorana-Weyl spinor of $\mathfrak{so}\,(10,2)$ is done by using the projection operators:

$$\mathcal{P}^{\alpha}{}_{\beta} = \frac{1}{2}(1+\Gamma_1\Gamma_2\Gamma_3\Gamma_4)^{\alpha}{}_{\beta} \qquad \mathcal{P}^{\dot\alpha}{}_{\dot\beta} = \frac{1}{2}(1+\Gamma_1\Gamma_2\Gamma_3\Gamma_4)^{\dot\alpha}{}_{\dot\beta} \tag{6.17}$$

where we assumed we compactify first 4 compact directions. This projection will reduce number of spinor components down to 16. It is clear that projected spinors will have the same chirality as their ancestors:

$$\mathcal{P}\Gamma_5\cdots\Gamma_{12}\mathcal{P} = \mathcal{P}\Gamma_{13}\mathcal{P}\,. \tag{6.18}$$

This 16-component spinor would thus comprise 2 same chirality 8-components spinors of $\mathfrak{so}\,(6,2)$ satisfying symplectic Majorana-Weyl reality condition. Their R-group is $\mathfrak{su}\,(2)$ - part of the $\mathfrak{so}\,(4)$ of the transverse directions that leaves the projection operator invariant. Thus the relevant 5-graded decomposition of $\mathfrak{e}_{7(-5)}$

$$\mathfrak{e}_{7(-5)} = \tilde{\mathbf{1}} \oplus \tilde{\mathbf{32}} \oplus [\mathfrak{so}^*(12)\oplus\mathfrak{so}(1,1)] \oplus \mathbf{32} \oplus \mathbf{1}$$

reads

$$\begin{aligned}\mathfrak{e}_{7(-5)} &= \mathbf{1} \oplus \begin{pmatrix}(\mathbf{2},\mathbf{1},\mathbf{8}_v)\\(\mathbf{1},\mathbf{2},\mathbf{8}_c)\end{pmatrix} \oplus \\ &\oplus \left[\Delta \oplus \left[\begin{array}{c}(\mathbf{2},\mathbf{2},\mathbf{8}_s)\\ \mathfrak{sp}\,(2,\mathbb{R})\oplus\mathfrak{su}\,(2)\oplus\mathfrak{so}\,(6,2)\end{array}\right]\right] \oplus \begin{pmatrix}(\mathbf{2},\mathbf{1},\mathbf{8}_v)\\(\mathbf{1},\mathbf{2},\mathbf{8}_c)\end{pmatrix} \oplus \mathbf{1}\,.\end{aligned} \tag{6.19}$$

Let $\xi^{i,\alpha}$ be an $\mathfrak{su}\,(2)$ doublet of $\mathfrak{so}\,(6,2)$ chiral spinors (symplectic Majorana-Weyl spinor) with $a,b,\ldots=1,2$ and $\alpha,\beta,\ldots=1,2,\ldots,8$. Then one can realize the Lie algebra of $\mathfrak{so}^*(12)$ of grade zero subspace as

$$\begin{aligned}M_{\mu\nu} &= \eta_{\mu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\mu,a}} - \xi^{i,\alpha}(\Gamma_{\mu\nu})^{\beta}{}_{\alpha}\frac{\partial}{\partial\xi^{i,\beta}}\\ J_{ab} &= \epsilon_{ac}X^{\mu,c}\frac{\partial}{\partial X^{\mu,b}} + \epsilon_{bc}X^{\mu,c}\frac{\partial}{\partial X^{\mu,a}}\\ L_{ij} &= \epsilon_{ik}\xi^{k,\alpha}\frac{\partial}{\partial\xi^{j,\alpha}} + \epsilon_{jk}\xi^{k,\alpha}\frac{\partial}{\partial\xi^{i,\alpha}}\\ Q_{ia\dot\alpha} &= \epsilon_{ab}X^{\mu,b}(\Gamma_\mu)^{\beta}{}_{\dot\alpha}\frac{\partial}{\partial\xi^{i,\beta}} - \epsilon_{ij}\psi^{j,\beta}\,(C\Gamma_\mu)_{\beta\dot\alpha}\,\eta^{\mu\nu}\frac{\partial}{\partial X^{\nu,a}}\end{aligned} \tag{6.20}$$

where $C^t = C$ and $\mu,\nu,\ldots = 1,2,\ldots,8$. Generators M,J,L,Q form $\mathfrak{so}^*\,(12)$ algebra:

$$\begin{aligned}[M_{\mu\nu},Q_{ia\dot\alpha}] &= Q_{ia\dot\beta}\,(\Gamma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\\ [L_{ij},L_{km}] &= \epsilon_{kj}L_{im}+\epsilon_{ki}L_{jm}+\epsilon_{mj}L_{ik}+\epsilon_{mi}L_{jk}\\ [J_{ab},Q_{ic\dot\alpha}] &= \epsilon_{cb}Q_{ia\dot\alpha}+\epsilon_{ca}Q_{ib\dot\alpha}\\ \left[Q_{ia\dot\alpha},Q_{jb\dot\beta}\right] &= \epsilon_{ij}\epsilon_{ab}\,(C\Gamma_{\mu\nu})_{\dot\alpha\dot\beta}\,M^{\mu\nu}+\epsilon_{ij}C_{\dot\alpha\dot\beta}J_{ab}+\epsilon_{ab}C_{\dot\alpha\dot\beta}L_{ij}\end{aligned} \tag{6.21}$$

corresponding to the decomposition

$$\mathfrak{so}^*\,(12) \supset \mathfrak{so}^*\,(8)\oplus\mathfrak{so}^*\,(4) \equiv \mathfrak{so}\,(6,2)\oplus\mathfrak{su}\,(2)\oplus\mathfrak{sp}\,(2,\mathbb{R})\,. \tag{6.22}$$

The fourth order invariant of $\mathfrak{so}^*(12)$ in the above basis is given by

$$\begin{aligned}\mathcal{I}_4 = &\eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd}X^{\mu,a}X^{\nu,b}X^{\rho,c}X^{\tau,d} - 2\epsilon_{ij}\epsilon_{ab}X^{\mu,a}X^{\nu,b}\xi^{i,\alpha}\left(C\Gamma_{\mu\nu}\right)_{\alpha\beta}\xi^{j,\beta} + \\ &+\frac{1}{4}\xi^{i,\alpha}\left(C\Gamma_{\mu\nu}\right)_{\alpha\beta}\xi^{j,\beta}\xi^{k,\gamma}\left(C\Gamma^{\mu\nu}\right)_{\gamma\delta}\xi^{l,\delta}\epsilon_{ij}\epsilon_{kl}\,. \end{aligned} \tag{6.23}$$

We can now write generators of $\mathfrak{e}_{7(-5)}$, starting with negative grade generators

$$\begin{aligned} K_- &= \frac{\partial}{\partial x} \qquad U_{i,\alpha} = \frac{\partial}{\partial \xi^{i,\alpha}} + \epsilon_{ij}C_{\alpha\beta}\xi^{j,\beta}\frac{\partial}{\partial x} \\ U_{\mu,a} &= \frac{\partial}{\partial X^{\mu,a}} - \eta_{\mu,\nu}\epsilon_{ab}X^{\nu,b}\frac{\partial}{\partial x}\,. \end{aligned} \tag{6.24}$$

Positive grade $+2$ generator K_+ is

$$\begin{aligned} K_+ = &\frac{1}{2}\left(2x^2 - \mathcal{I}_4\right)\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} + \\ &+\frac{1}{4}\epsilon^{ij}\frac{\partial \mathcal{I}_4}{\partial \xi^{i,\alpha}}\left(C^{-1}\right)^{\alpha\beta}\frac{\partial}{\partial \xi^{j,\beta}} + x\,X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} + x\,\xi^{i,\alpha}\frac{\partial}{\partial \xi^{i,\alpha}} \end{aligned} \tag{6.25}$$

Commutation relations of $\mathfrak{g}^{-1}$ and $\mathfrak{g}^{+1}$ specific to $6+2=8$ dimensions are:

$$\begin{aligned} \left[U_{i,\alpha},\tilde{U}_{j,\beta}\right] &= \epsilon_{ij}\left(C\Gamma_{\mu\nu}\right)_{\alpha\beta}M^{\mu\nu} + C_{\alpha\beta}L_{ij} - \epsilon_{ij}C_{\alpha\beta}\Delta \\ \left[U_{i,\alpha},\tilde{U}_{\mu,a}\right] &= -\left(C\Gamma_\mu C^{-1}\right)_\alpha{}^{\dot\alpha}Q_{ia\dot\alpha}\,. \end{aligned} \tag{6.26}$$

Grade $+1$ generators have the following form:

$$\begin{aligned} \tilde{U}_{\mu,a} = &-\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\frac{\partial}{\partial y} - \eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\,y\frac{\partial}{\partial y} + y\frac{\partial}{\partial X^{\mu,a}} - \\ &-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}\partial X^{\nu,b}}\eta^{\nu\rho}\epsilon^{bc}\frac{\partial}{\partial X^{\rho,c}} + \frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}\xi^{i,\alpha}}\epsilon^{ij}\left(C^{-1}\right)^{\alpha\beta}\frac{\partial}{\partial \xi^{j,\beta}} - \\ &-\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \xi^{i,\alpha}\frac{\partial}{\partial \xi^{i,\alpha}}\right) \end{aligned} \tag{6.27}$$

$$\begin{aligned} \tilde{U}_{i,\alpha} = &-\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial \xi^{i,\alpha}}\frac{\partial}{\partial y} + C_{\alpha\beta}\epsilon_{ij}\xi^{j,\beta}\,y\frac{\partial}{\partial y} + y\frac{\partial}{\partial \xi^{i,\alpha}} + \\ &+\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial \xi^{i,\alpha}\partial \xi^{j,\beta}}\left(C^{-1}\right)^{\beta\gamma}\epsilon^{jk}\frac{\partial}{\partial \xi^{k,\gamma}} - \frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial \xi^{i,\alpha}X^{\mu,a}}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} + \\ &+C_{\alpha\beta}\epsilon_{ij}\xi^{j,\beta}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \xi^{i,\alpha}\frac{\partial}{\partial \xi^{i,\alpha}}\right). \end{aligned} \tag{6.28}$$

6.3 Geometric realization of the quasiconformal group $\mathrm{E}_{6(2)}$

Truncation to $\mathfrak{e}_{6(2)}$ is done by further dimensional reduction from $d=6$ to $d=4$. Projecting spinors is done in a similar way and results in breaking R-symmetry algebra to $\mathfrak{u}(1)$. The resulting 5-graded decomposition of $\mathfrak{e}_{6(2)}$ is:

$$\mathfrak{e}_{6(2)} = \tilde{\mathbf{1}} \oplus \tilde{\mathbf{20}} \oplus \left[\mathfrak{su}(3,3) \oplus \mathfrak{so}(1,1)\right] \oplus \mathbf{20} \oplus \mathbf{1} \tag{6.29}$$

$$\mathfrak{e}_{6(2)} = \mathbf{1} \oplus \begin{pmatrix} (\mathbf{2}, \mathbf{6}_v) \\ (\mathbf{1}, \mathbf{4}_c)^+ \\ (\mathbf{1}, \mathbf{4}_s)^- \end{pmatrix} \oplus$$

$$\oplus \left[\Delta \oplus \left[\begin{array}{c} (\mathbf{2}, \mathbf{4}_s)^+ \oplus (\mathbf{2}, \mathbf{4}_c)^- \\ \mathfrak{sp}\,(2, \mathbb{R}) \oplus \mathfrak{u}\,(1) \oplus \mathfrak{so}\,(4,2) \end{array} \right] \right] \oplus \begin{pmatrix} (\mathbf{2}, \mathbf{6}_v) \\ (\mathbf{1}, \mathbf{4}_c)^+ \\ (\mathbf{1}, \mathbf{4}_s)^- \end{pmatrix} \oplus \mathbf{1} \tag{6.30}$$

where $+$ and $-$ refer to ± 1 charges of $\mathfrak{u}\,(1)$. Let ζ^α be a chiral spinor of $\mathfrak{so}\,(4,2)$ and $\zeta^{\dot\alpha}$ the corresponding anti-chiral spinor. We shall combine these two chiral spinors into one Majorana spinor ψ^A of $\mathfrak{so}\,(4,2)$. The decomposition of the The generators of $\mathfrak{su}(3,3)$ in $\mathfrak{g}^0$ subspace read as follows

$$\begin{aligned} M_{\mu\nu} &= \eta_{\mu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\mu,a}} - \psi^A (\Gamma_{\mu\nu})^A{}_B \frac{\partial}{\partial \psi^B} \\ H &= \zeta^\alpha \frac{\partial}{\partial \zeta^\alpha} - \zeta^{\dot\alpha} \frac{\partial}{\partial \zeta^{\dot\alpha}} = \psi^A (\Gamma_7)^B{}_A \frac{\partial}{\partial \psi^B} \\ Q_{a,A} &= \epsilon_{ab} X^{\mu,a} (\Gamma_\mu)^B{}_A \frac{\partial}{\partial \psi^B} + \eta^{\mu\nu} \psi^B \left(C \Gamma_\mu \Gamma_7 \right)_{BA} \frac{\partial}{\partial X^{\nu,a}} \end{aligned}$$

while J_{ab} is defined as before and the charge conjugation matrix is now symmetric $C^t = C$. These generators of $\mathfrak{su}\,(3,3)$ satisfy the commutation relations

$$\begin{aligned} [Q_{a,A}, Q_{b,B}] &= \frac{3}{2} \epsilon_{ab} C_{AB} H - \epsilon_{ab} \left(C \Gamma_{\mu\nu} \right)_{AB} M^{\mu\nu} - \left(C \Gamma_7 \right)_{AB} J_{ab} \\ [H, Q_{a,A}] &= (\Gamma_7)^B{}_A Q_{a,B} \,. \end{aligned} \tag{6.31}$$

The chiral components of the generators of $Q_{a,A}$ are given by

$$\begin{aligned} Q_{a,\alpha} &= \epsilon_{ab} X^{\mu,b} (\Gamma_\mu)^{\dot\alpha}{}_\alpha \frac{\partial}{\partial \zeta^{\dot\alpha}} + \eta^{\mu\nu} \zeta^\beta \left(C \Gamma_\mu \right)_{\beta\alpha} \frac{\partial}{\partial X^{\nu,a}} \\ Q_{a,\dot\alpha} &= \epsilon_{ab} X^{\mu,b} (\Gamma_\mu)^{\alpha}{}_{\dot\alpha} \frac{\partial}{\partial \zeta^{\alpha}} - \eta^{\mu\nu} \zeta^{\dot\beta} \left(C \Gamma_\mu \right)_{\dot\beta\dot\alpha} \frac{\partial}{\partial X^{\nu,a}} \end{aligned}$$

The 4-th order invariant of $\mathfrak{su}\,(3,3)$ written in terms of X and ψ reads as follows:

$$\begin{aligned} \mathcal{I}_4 &= \eta_{\mu\nu} \eta_{\rho\tau} \epsilon_{ac} \epsilon_{bd} X^{\mu,a} X^{\nu,b} X^{\rho,c} X^{\tau,d} - 2 X^{\mu,a} \epsilon_{ab} X^{\nu,b} \psi^A \psi^B \left(C \Gamma_{\mu\nu} \right)_{AB} \\ &+ \frac{1}{3} \eta^{\mu\nu} \eta^{\rho\tau} \psi^A \psi^B \left(C \Gamma_{\mu\rho} \right)_{AB} \psi^E \psi^F \left(C \Gamma_{\nu\tau} \right)_{EF} \,. \end{aligned} \tag{6.32}$$

The spinorial generators of $\mathrm{E}_{6(2)}$ belonging to $\mathfrak{g}^{-1}$ subspace are realized as

$$U_A = \frac{\partial}{\partial \psi^A} + \left(C \Gamma_7 \right)_{AB} \psi^B \frac{\partial}{\partial y} \tag{6.33}$$

which commute into the grade -2 generator K_-

$$[U_A, U_B] = 2 \left(C \Gamma_7 \right)_{AB} K_- \,. \tag{6.34}$$

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The commutators of the generators $Q_{a,A}$ belonging to grade zero subspace with those in grade -1 subspace read as

$$\begin{aligned} [Q_{a,A},U_B] &= -\left(C\Gamma_\mu\Gamma_7\right)_{AB}\eta^{\mu\nu}U_{\nu,a} \\ [Q_{a,A},U_{\mu,b}] &= -\epsilon_{ab}U_B(\Gamma_\mu)^B{}_A\,. \end{aligned} \tag{6.35}$$

Commutation relations of the form $\left[\mathfrak{g}^{-1},\mathfrak{g}^{+1}\right]\subset\mathfrak{g}^0$ are

$$\begin{aligned} \left[U_A,\tilde{U}_B\right] &= -\frac{3}{2}\left(C\Gamma_7\right)_{AB}H - C_{AB}\Delta + \left(C\Gamma_{\mu\nu}\right)_{AB}M^{\mu\nu} \\ \left[U_A,\tilde{U}_{\mu,a}\right] &= \left(C\Gamma_\mu C^{-1}\right)_A{}^B Q_{a,B} \\ \left[\tilde{U}_A,U_{\mu,a}\right] &= -\left(C\Gamma_\mu C^{-1}\right)_A{}^B Q_{a,B}\,. \end{aligned} \tag{6.36}$$

Explicit expressions for positive grade generators are as follows:

$$\begin{aligned} \tilde{U}_{\mu,a} = &-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial X^{\mu,a}}\frac{\partial}{\partial y} - \eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}y\frac{\partial}{\partial y} + y\frac{\partial}{\partial X^{\mu,a}} - \\ &-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}X^{\nu,b}}\eta^{\nu\rho}\epsilon^{bc}\frac{\partial}{\partial X^{\rho,c}} - \frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}\psi^A}\left(\Gamma_7C^{-1}\right)^{AB}\frac{\partial}{\partial\psi^B} - \\ &-\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \psi^A\frac{\partial}{\partial\psi^A}\right) \end{aligned} \tag{6.37}$$

$$\begin{aligned} \tilde{U}_A = &-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial\psi^A}\frac{\partial}{\partial y} - \left(C\Gamma_7\right)_{AB}\psi^B y\frac{\partial}{\partial y} + y\frac{\partial}{\partial\psi^A} - \\ &-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial\psi^A\psi^B}\left(\Gamma_7C^{-1}\right)^{BC}\frac{\partial}{\partial\psi^C} - \frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}\psi^A}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} - \\ &-\left(C\Gamma_7\right)_{AB}\psi^B\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \psi^C\frac{\partial}{\partial\psi^C}\right) \end{aligned} \tag{6.38}$$

$$\begin{aligned} K_+ = &\ \frac{1}{2}\left(2y^2-\mathcal{I}_4\right)\frac{\partial}{\partial y} + y\left(X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} + \psi^A\frac{\partial}{\partial\psi^A}\right) - \\ &-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial X^{\mu,a}}\epsilon^{ab}\eta^{\mu\nu}\frac{\partial}{\partial X^{\nu,b}} - \frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial\psi^A}\left(\Gamma_7C^{-1}\right)^{AB}\frac{\partial}{\partial\psi^B}\,. \end{aligned} \tag{6.39}$$

6.4 Geometric realization of the quasiconformal group $F_{4(4)}$

Further truncation to $\mathfrak{f}_{4(4)}$ is performed by reducing from $d=4$ to $d=3$. The 5-grading in this case is

$$\mathfrak{f}_{4(4)} = \tilde{\mathbf{1}}\oplus\tilde{\mathbf{14}}\oplus\left[\mathfrak{sp}(6,\mathbb{R})\oplus\mathfrak{so}(1,1)\right]\oplus\mathbf{14}\oplus\mathbf{1} \tag{6.40}$$

$$\begin{aligned} \mathfrak{f}_{4(4)} = &\ \mathbf{1}\oplus\begin{pmatrix}(\mathbf{2},\mathbf{5})\\(\mathbf{1},\mathbf{4})\end{pmatrix}\oplus \\ &\oplus\left[\Delta\oplus\left[\begin{matrix}(\mathbf{2},\mathbf{4})\\ \mathfrak{sp}\left(2,\mathbb{R}\right)\oplus\mathfrak{so}\left(3,2\right)\end{matrix}\right]\right]\oplus\begin{pmatrix}(\mathbf{2},\mathbf{5})\\(\mathbf{1},\mathbf{4})\end{pmatrix}\oplus\mathbf{1}\,. \end{aligned} \tag{6.41}$$

We use the same notations for spinors as above, assuming that now $A=1,\ldots,4$.

$$\begin{aligned} M_{\mu\nu} &= \eta_{\mu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\mu,a}} - \psi^A(\Gamma_{\mu\nu})^A{}_B\frac{\partial}{\partial\psi^B} \\ Q_{a,A} &= \epsilon_{ab}X^{\mu,a}(\Gamma_\mu)^B{}_A\frac{\partial}{\partial\psi^B} + \eta^{\mu\nu}\psi^B\left(C\Gamma_\mu\right)_{BA}\frac{\partial}{\partial X^{\nu,a}} \end{aligned}$$

where $C^t = -C$. The generators $Q_{a,A}$ close into $\mathfrak{sp}\,(2,\mathbb{R}) \oplus \mathfrak{so}\,(3,2)$ as follows:

$$[Q_{a,A}, Q_{b,B}] = \epsilon_{ab} \left(C\Gamma_{\mu\nu}\right)_{AB} M^{\mu\nu} + C_{AB} J_{ab} \tag{6.42}$$

and transform under $\mathfrak{sp}\,(2,\mathbb{R}) \oplus \mathfrak{so}\,(3,2)$ in the $(\mathbf{2},\mathbf{4})$ representation. The generators $Q_{a,A}, M^{\mu\nu}$ and J_{ab} form the $\mathfrak{sp}(6,\mathbb{R})$ subalgebra.

Generators $K_- \in \mathfrak{g}^{-2}$ and $U_{\mu,a} \in \mathfrak{g}^{-1}$ are as above and spinorial generators of $\mathfrak{g}^{-1}$ are given by

$$U_A = \frac{\partial}{\partial \psi^A} + (C)_{AB}\, \psi^B \frac{\partial}{\partial y}\,. \tag{6.43}$$

Spinorial generators form an Heisenberg subalgebra with charge conjugation matrix C serving as symplectic metric:

$$[U_A, U_B] = -2 C_{AB} K_-\,. \tag{6.44}$$

The generators Q act on $\mathfrak{g}^{-1}$ subspace as follows

$$\begin{aligned} [Q_{a,A}, U_B] &= \left(C\Gamma_\mu\right)_{AB} \eta^{\mu\nu} U_{\nu,a} \\ [Q_{a,A}, U_{\mu,b}] &= -\epsilon_{ab} (\Gamma_\mu)^B{}_A U_B\,. \end{aligned} \tag{6.45}$$

Quartic invariant $\mathcal{I}_4$ of $\mathfrak{sp}(6,\mathbb{R})$ in $\mathfrak{sp}\,(2,\mathbb{R}) \oplus \mathfrak{so}\,(3,2)$ basis is given by

$$\mathcal{I}_4 = \eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd} X^{\mu,a} X^{\nu,b} X^{\rho,c} X^{\tau,d} - 2 X^{\mu,a} \epsilon_{ab} X^{\nu,b} \psi^A \psi^B \left(C\Gamma_{\mu\nu}\right)_{AB}\,. \tag{6.46}$$

Notice that the quartic term involving purely spinorial coordinates ,present in previous cases, now vanishes, since there is no symmetric rank 4 invariant tensor of $\mathfrak{so}\,(3,2) \simeq \mathfrak{sp}\,(4,\mathbb{R})$ over its spinorial representation space. Then the positive grade generators are

$$\begin{aligned} K_+ &= \frac{1}{2}\left(2y^2 - \mathcal{I}_4\right)\frac{\partial}{\partial y} + y\left(X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} + \psi^A \frac{\partial}{\partial \psi^A}\right) - \\ &\quad - \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}} \epsilon^{ab}\eta^{\mu\nu}\frac{\partial}{\partial X^{\nu,b}} + \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial \psi^A}\left(C^{-1}\right)^{AB}\frac{\partial}{\partial \psi^B} \end{aligned} \tag{6.47}$$

$$\begin{aligned} \tilde{U}_{\mu,a} &= -\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\frac{\partial}{\partial y} - \eta_{\mu\nu}\epsilon_{ab} X^{\nu,b} y \frac{\partial}{\partial y} + y \frac{\partial}{\partial X^{\mu,a}} - \\ &\quad - \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a} X^{\nu,b}} \eta^{\nu\rho}\epsilon^{bc}\frac{\partial}{\partial X^{\rho,c}} + \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\psi^A}\left(C^{-1}\right)^{AB}\frac{\partial}{\partial \psi^B} - \\ &\quad - \eta_{\mu\nu}\epsilon_{ab} X^{\nu,b}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \psi^A \frac{\partial}{\partial \psi^A}\right) \end{aligned} \tag{6.48}$$

$$\begin{aligned} \tilde{U}_A &= -\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial \psi^A}\frac{\partial}{\partial y} + C_{AB}\psi^B y \frac{\partial}{\partial y} + y\frac{\partial}{\partial \psi^A} + \\ &\quad + \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial \psi^A \psi^B}\left(C^{-1}\right)^{BC}\frac{\partial}{\partial \psi^C} - \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\psi^A}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} + \\ &\quad + (C)_{AB}\,\psi^B\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \psi^C \frac{\partial}{\partial \psi^C}\right). \end{aligned} \tag{6.49}$$

Commutation relations of generators belonging to $\mathfrak{g}^{-1}$ and to $\mathfrak{g}^{+1}$ are

$$\left[U_A, \tilde{U}_B\right] = \left(C\Gamma_{\mu\nu}\right)_{AB} M^{\mu\nu} - C_{AB}\Delta$$

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$$\begin{aligned}\left[U_A, \tilde{U}_{\mu,a}\right] &= -\left(C\Gamma_\mu C^{-1}\right)_A{}^B Q_{a,B} \\ \left[\tilde{U}_A, U_{\mu,a}\right] &= \left(C\Gamma_\mu C^{-1}\right)_A{}^B Q_{a,B} \,. \end{aligned} \tag{6.50}$$

7. Minimal unitary realizations of the quasiconformal groups SO$(d+2,4)$

In our earlier work [10] we constructed the minimal unitary representations of the exceptional groups discussed in the previous section. In this section we will construct the minimal unitary representations of the groups SO$\,(d+2,4)$, corresponding to the quantization of their geometric realizations as quasiconformal groups given in section 5 following methods of [28, 10]. Let X^μ and P_μ be canonical coordinates and momenta in $\mathbb{R}^{(2,d)}$:

$$[X^\mu, P_\nu] = i\delta^\mu_\nu \,. \tag{7.1}$$

In the notation of section 5 we identify $X^{\mu,a=1}$ to be coordinates X^μ, and $P_\mu = \eta_{\mu\nu} X^{\nu,a=2}$ to be conjugate momenta. Also let x be an additional "cocycle" coordinate and p be its conjugate momentum:

$$[x, p] = i \,. \tag{7.2}$$

The grade zero generators $(M_{\mu\nu}, J_\pm, J_0)$, grade -1 generators (U_μ, V^μ), grade -2 generator K_- and the 4-th order invariant $\mathcal{I}_4$ of the semisimple part of the grade zero subalgebra are realized as follows:

$$\begin{aligned} M_{\mu\nu} &= i\eta_{\mu\rho} X^\rho P_\nu - i\eta_{\nu\rho} X^\rho P_\mu \qquad J_0 = \frac{1}{2}\left(X^\mu P_\mu + P_\mu X^\mu\right) \\ U_\mu &= xP_\mu \quad V^\mu = xX^\mu \qquad J_- = X^\mu X^\nu \eta_{\mu\nu} \\ K_- &= \frac{1}{2}x^2 \qquad J_+ = P_\mu P_\nu \eta^{\mu\nu} \\ \mathcal{I}_4 &= \left(X^\mu X^\nu \eta_{\mu\nu}\right)\left(P_\mu P_\nu \eta^{\mu\nu}\right) + \left(P_\mu P_\nu \eta^{\mu\nu}\right)\left(X^\mu X^\nu \eta_{\mu\nu}\right) - \\ &\quad - \left(X^\mu P_\mu\right)\left(P_\nu X^\nu\right) - \left(P_\mu X^\mu\right)\left(X^\nu P_\nu\right) . \end{aligned} \tag{7.3}$$

Using the quartic invariant we define the grade $+2$ generator as

$$K_+ = \frac{1}{2}p^2 + \frac{1}{4\,y^2}\left(\mathcal{I}_4 + \frac{d^2+3}{2}\right) . \tag{7.4}$$

It is easy to verify that the generators $M_{\mu\nu}$ and $J_{0,\pm}$ satisfy the commutation relations of $\mathfrak{so}\,(d,2) \oplus \mathfrak{sp}\,(2,\mathbb{R})$

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\tau}] &= \eta_{\nu\rho} M_{\mu\tau} - \eta_{\mu\rho} M_{\nu\tau} + \eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\tau} M_{\mu\rho} \\ [J_0, J_\pm] &= \pm 2i J_\pm \qquad [J_-, J_+] = 4i J_0 \end{aligned} \tag{7.5}$$

under which coordinates X^μ and momenta P^μ transform as Lorentz vectors and form doublets of the symplectic group $Sp(2,\mathbb{R})$:

$$\begin{aligned} &[J_0, V^\mu] = -iV^\mu \quad [J_-, V^\mu] = 0 \qquad\quad [J_+, V^\mu] = -2i\eta^{\mu\nu} U_\nu \\ &[J_0, U_\mu] = +iU_\mu \quad [J_-, U_\mu] = 2i\eta_{\mu\nu} V^\nu \quad [J_+, U_\mu] = 0 \,. \end{aligned} \tag{7.6}$$

The generators in the subspace $\mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$ form a Heisenberg algebra

$$[V^\mu, U_\nu] = 2i\delta^\mu{}_\nu \, K_- \,. \tag{7.7}$$

Define the grade +1 generators as

$$\tilde{V}^\mu = -i\,[V^\mu, K_+] \qquad \tilde{U}_\mu = -i\,[U_\mu, K_+] \tag{7.8}$$

which explicitly read as follows

$$\begin{aligned}
\tilde{V}^\mu &= pX^\mu + \frac{1}{2}x^{-1}\left(P_\nu X^\lambda X^\rho + X^\lambda X^\rho P_\nu\right)\eta^{\mu\nu}\eta_{\lambda\rho} - \\
&\quad -\frac{1}{4}x^{-1}\left(X^\mu\left(X^\nu P_\nu + P_\nu X^\nu\right) + \left(X^\nu P_\nu + P_\nu X^\nu\right)X^\mu\right) \\
\tilde{U}_\mu &= pP_\mu - \frac{1}{2}x^{-1}\left(X^\nu P_\lambda P_\rho + P_\lambda P_\rho X^\nu\right)\eta_{\mu\nu}\eta^{\lambda\rho} + \\
&\quad +\frac{1}{4}x^{-1}\left(P_\mu\left(X^\nu P_\nu + P_\nu X^\nu\right) + \left(X^\nu P_\nu + P_\nu X^\nu\right)P_\mu\right).
\end{aligned} \tag{7.9}$$

Then one finds that the generators in $\mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$ subspace form an isomorphic Heisenberg algebra

$$\left[\tilde{V}^\mu, \tilde{U}_\nu\right] = 2i\delta^\mu{}_\nu K_+ \qquad V^\mu = i\left[\tilde{V}^\mu, K_-\right] \qquad U_\mu = i\left[\tilde{U}_\mu, K_-\right]. \tag{7.10}$$

Commutators $\left[\mathfrak{g}^{-1}, \mathfrak{g}^{+1}\right]$ close into $\mathfrak{g}^0$ as follows

$$\begin{aligned}
\left[U_\mu, \tilde{U}_\nu\right] &= i\eta_{\mu\nu} J_- \qquad \left[V^\mu, \tilde{V}^\nu\right] = i\eta^{\mu\nu} J_+ \\
\left[V^\mu, \tilde{U}_\nu\right] &= 2\eta^{\mu\rho} M_{\rho\nu} + i\delta^\mu{}_\nu\left(J_0 + \Delta\right) \\
\left[U_\mu, \tilde{V}^\nu\right] &= -2\eta^{\nu\rho} M_{\mu\rho} + i\delta^\nu{}_\mu\left(J_0 - \Delta\right)
\end{aligned} \tag{7.11}$$

where Δ is the generator that determines the 5-grading

$$\Delta = \frac{1}{2}\left(xp + px\right) \tag{7.12}$$

such that

$$[K_-, K_+] = i\Delta \qquad [\Delta, K_\pm] = \pm 2iK_\pm \tag{7.13}$$

$$[\Delta, U_\mu] = -iU_\mu \quad [\Delta, V^\mu] = -iV^\mu \quad \left[\Delta, \tilde{U}_\mu\right] = i\tilde{U}_\mu \quad \left[\Delta, \tilde{V}^\mu\right] = i\tilde{V}^\mu \,. \tag{7.14}$$

The quadratic Casimir operators of subalgebras $\mathfrak{so}\,(d,2)$, $\mathfrak{sp}\,(2,\mathbb{R})_J$ of grade zero subspace and $\mathfrak{sp}\,(2,\mathbb{R})_K$ generated by $K_\pm$ and Δ are

$$\begin{aligned}
M_{\mu\nu}M^{\mu\nu} &= -\mathcal{I}_4 - 2\,(d+2) \\
J_-J_+ + J_+J_- - 2\,(J_0)^2 &= \mathcal{I}_4 + \frac{1}{2}\,(d+2)^2 \\
K_-K_+ + K_+K_- - \frac{1}{2}\Delta^2 &= \frac{1}{4}\mathcal{I}_4 + \frac{1}{8}\,(d+2)^2 \,.
\end{aligned} \tag{7.15}$$

Note that they all reduce to $\mathcal{I}_4$ modulo some additive constants. Noting also that

$$\left(U_\mu \tilde{V}^\mu + \tilde{V}^\mu U_\mu - V^\mu \tilde{U}_\mu - \tilde{U}_\mu V^\mu\right) = 2\mathcal{I}_4 + (d+2)(d+6) \tag{7.16}$$

we conclude that there exists a family of degree 2 polynomials in the enveloping algebra of $\mathfrak{so}(d+2,4)$ that degenerate to a c-number for the minimal unitary realization, in accordance with Joseph's theorem [29]:

$$\begin{aligned} M_{\mu\nu}M^{\mu\nu} + \kappa_1 \left(J_- J_+ + J_+ J_- - 2(J_0)^2\right) + \\ +4\kappa_2 \left(K_- K_+ + K_+ K_- - \frac{1}{2}\Delta^2\right) - \\ -\frac{1}{2}(\kappa_1 + \kappa_2 - 1)\left(U_\mu \tilde{V}^\mu + \tilde{V}^\mu U_\mu - V^\mu \tilde{U}_\mu - \tilde{U}_\mu V^\mu\right) = \\ = \frac{1}{2}(d+2)(d+2-4(\kappa_1+\kappa_2)). \end{aligned} \tag{7.17}$$

The quadratic Casimir of $\mathfrak{so}(d+2,4)$ corresponds to the choice $2\kappa_1 = 2\kappa_2 = -1$ in (7.17). Hence the eigenvalue of the quadratic Casimir for the minimal unitary representation is equal to $\frac{1}{2}(d+2)(d+6)$.

Acknowledgments

This work was supported in part by the National Science Foundation under grant number PHY-0245337. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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